POLAR DESCRIPTION OF MOTION:

1/16/2026

$\hat{e}_r$ unit vector in radial direction (think think about a telescopic rod moving away from- or closer to observer)

$\hat{e}_\theta$ unit vector in polar direction $\perp$ to $\hat{e}_r$ pointing in positive θ direction

e.g.: Swing ride The position of a rider is defined by $\vec{r}$ from O and θ from $0°$

In polar kinematics, we write the position vector of P:

$$\boxed{\vec{r} = r\,\hat{e}_r}$$

The velocity of P is

$$\vec{v} = \frac{d}{dt}(r\,\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\,\frac{d}{dt}[\hat{e}_r]$$

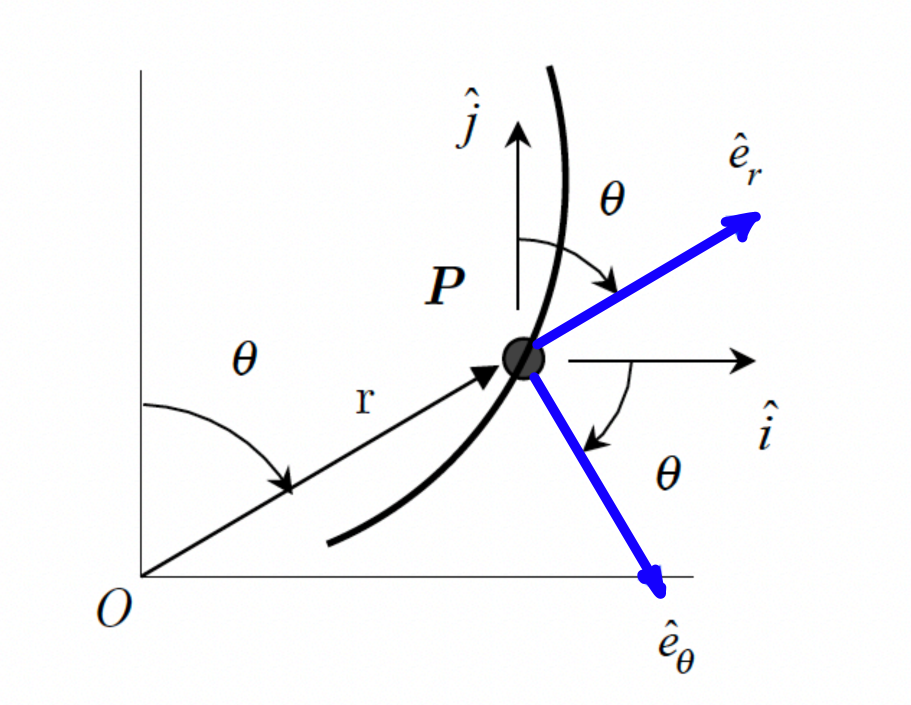
Due to $\hat{e}_r$'s dependency of θ, the chain rule is used:

$$\vec{v} = \dot{r}\,\hat{e}_r + r\,\frac{d\hat{e}_r}{d\theta}\cdot\frac{d\theta}{dt} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\frac{d}{d\theta}(\hat{e}_r)$$

How do $\hat{e}_r$ and $\hat{e}_\theta$ change with respect to θ?

$$\hat{e}_r = \sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \Rightarrow \frac{d}{d\theta}(\hat{e}_r) = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} = \hat{e}_\theta$$

$$\hat{e}_\theta = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} \Rightarrow \frac{d}{d\theta}(\hat{e}_\theta) = -\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath} = -\hat{e}_r$$



Let us write now the equation of the velocity vector:

$$\boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}$$

Next, let's use the product rule to obtain an expression for the acceleration vector:

$$\vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \dot{r}(\dot{\theta}\hat{e}_\theta) + r\frac{d}{dt}(\dot{\theta}\hat{e}_\theta)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \dot{r}(\dot{\theta}\hat{e}_\theta) + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{e}_\theta)$$

Now, let's obtain $\frac{d}{dt}(\hat{e}_r)$ and $\frac{d}{dt}(\hat{e}_\theta)$

$$\frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$

$$= \cos\theta\,\dot{\theta}\,\hat{\imath} - \sin\theta\,\dot{\theta}\,\hat{\jmath}$$

$$= \dot{\theta}\underbrace{(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})}_{\hat{e}_\theta}$$

$$\frac{d}{dt}(\hat{e}_r) = \dot{\theta}\,\hat{e}_\theta$$

We are computing $\frac{d}{dt}$ here.
Previously, we did $\frac{d}{d\theta}$

Similarly:

$$\frac{d}{dt}(\hat{e}_\theta) = \frac{d}{dt}(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$= -\sin\theta\cdot\dot{\theta}\,\hat{\imath} - \cos\theta\cdot\dot{\theta}\,\hat{\jmath}$$

$$= \dot{\theta}\,\underbrace{(-\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})}_{\hat{e}_r}$$

$$\frac{d}{dt}(\hat{e}_\theta) = -\dot{\theta}\,\hat{e}_r$$

Therefore, factoring everything in, we have:

$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta}$$

The $(\ddot{r} - r\dot{\theta}^2)$ and $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ terms have a more unintuitive physical interpretation. Their meaning is geometric rather than Kinematic

Polar description: more insights.

Recall:

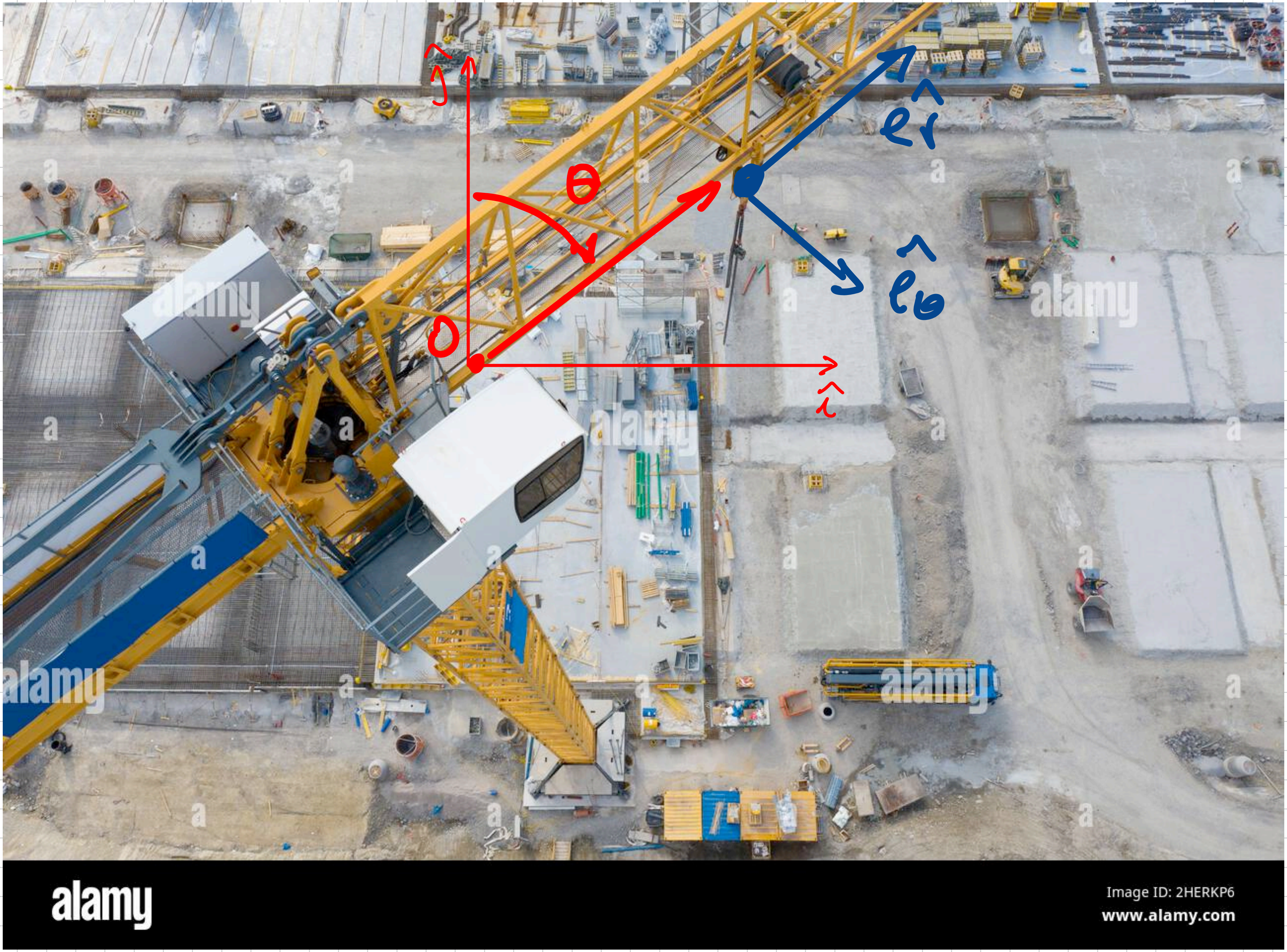
$$\boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta.}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta}$$

The values of the components $\hat{e}_r$ & $\hat{e}_\theta$ depend on your choice of point O $\Rightarrow$ You need to carefully establish your choice of O at the beginning of the solution of the problem.

When the path of P is $r(\theta)$, you need to use the chain rule to find $\dot{r}$ and $\ddot{r}$ in terms of $\dot{\theta}$ and $\ddot{\theta}$

In many applications involving observer motion, polar kinematics are very useful (e.g.: the operator of a construction crane)


$\hat{j}$
θ
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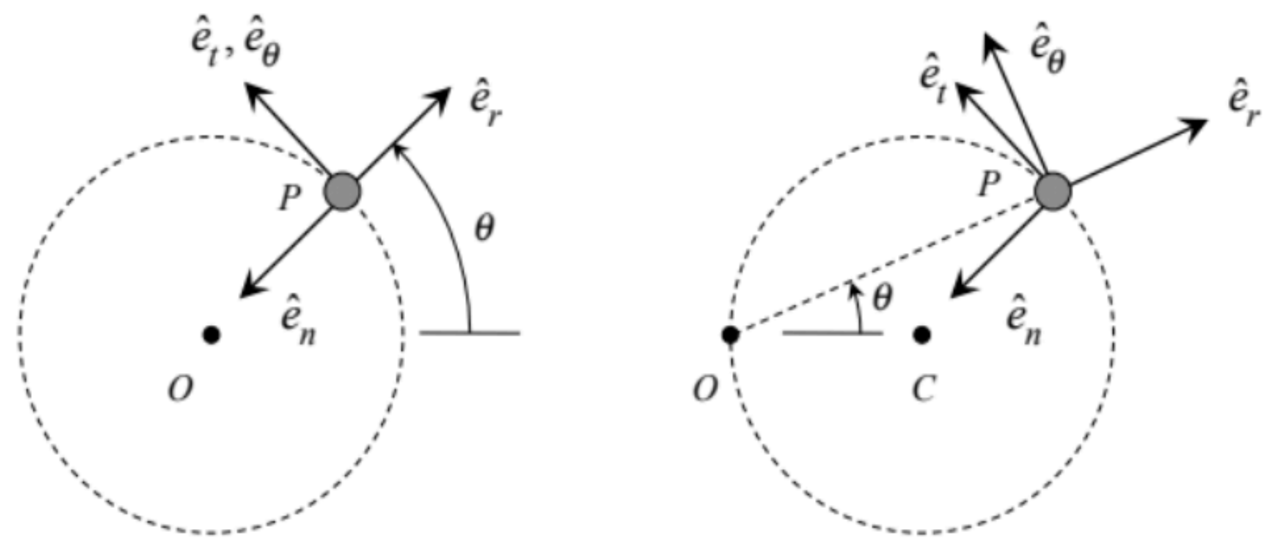
Challenge!

Is there an instance where ($\hat{e}_t$ & $\hat{e}_n$) are aligned with ($\hat{e}_r$ & $\hat{e}_\theta$)?

Yes!

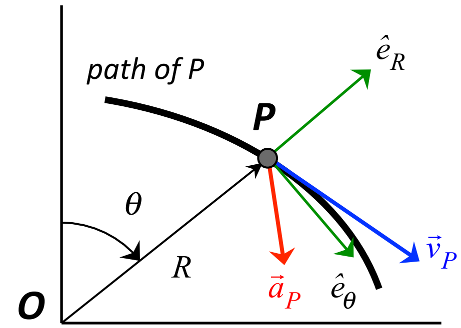
Recall that $\hat{e}_t$ & $\hat{e}_n$ are defined by the path and $\hat{e}_r$ & $\hat{e}_\theta$ are defined by our choice of of observer O and the position r relative to O

There is some alignment in the special case where the path is circular and the observer is at the center.



Summary: Particle Kinematics – Polar Description

1. *PROBLEM*: Motion of a point described in polar coordinates, *R* and θ.
2. *FUNDAMENTAL EQUATIONS*:



$$\vec{v}_P = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = velocity\ of\ P$$

$$\vec{a}_P = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta = acceleration\ of\ P$$

where $\hat{e}_R$ and $\hat{e}_\theta$ are the radial and transverse unit vectors.

3. *OBSERVATIONS: In regard to the polar description kinematics, we see*
   - *You* are free to choose the observation point O.
   - $\hat{e}_R$ *always* points OUTWARD from O to P. $\hat{e}_\theta$ is perpendicular to $\hat{e}_R$ and in direction of increasing θ.
   - Polar description is useful for problems with observers or rotations about fixed axes.
   - Do not confuse the unit radial vector $\hat{e}_R$ with the unit normal vector $\hat{e}_n$.

$\ddot{\theta} = 0$

Additional lecture Example 1.4

Given: A rotating and telescoping robotic arm is gripping a small sphere P in its end effector. The arm is rotating counterclockwise with a constant angular speed of $\dot{\theta}$. The arm is extending such that the radial distance from O to P is related to the rotation angle θ by the following equation:

$$r(\theta) = R_0 + R_1 \cos 2\theta$$

where r and θ are given in terms of meters and radians, respectively.

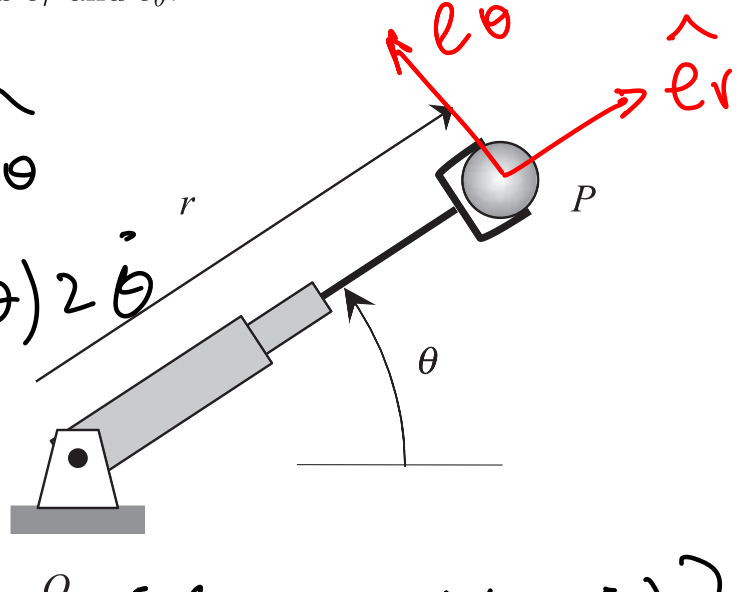
Find: Determine the velocity and acceleration of the sphere P. Write your answers as vectors in terms of the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$.

Solution

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\dot{r}(\theta) = 0 + R_1(-\sin 2\theta)2\dot{\theta}$$

$$\dot{r} = -2R_1\dot{\theta}\sin 2\theta$$



$$\ddot{r} = -2R_1\left[\ddot{\theta}\sin 2\theta + \dot{\theta}(-\cos 2\theta)(2)(\dot{\theta})\right]$$

Use the following parameters in your analysis: $R_0 = 2$ m, $R_1 = 0.5$ m, $\theta = \pi/2$ rad and $\dot{\theta} = 2$ rad/s.

$$\ddot{r} = -2R_1\left(\ddot{\theta}\sin 2\theta - 2\dot{\theta}^2\cos 2\theta\right)$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$= -2R_1\dot{\theta}\sin 2\theta\;\hat{e}_r + \overbrace{(R_0 + R_1\cos 2\theta)}^{r(\theta)}\,\underline{\dot{\theta}}\;\hat{e}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\cancelto{0}{\ddot{\theta}} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

$$= \left[-2R_1(\ddot{\theta}\sin 2\theta + 2\dot{\theta}^2\cos 2\theta) - (R_0 + R_1\cos 2\theta)\dot{\theta}\right]\hat{e}_r$$

$$+\left[-4R_1\dot{\theta}^2\sin 2\theta\right]\;\hat{e}_\theta$$

Additional lecture Example 1.3

Given: Particle P travels along an elliptical path shown with $\dot{\theta}$ = constant. $\ddot{\theta} = 0$

Find: For the position of P corresponding to $\theta = \pi/2$:

(a) Determine $\dot{R}$ and $\ddot{R}$. It is recommended that you use implicit differentiation for this.

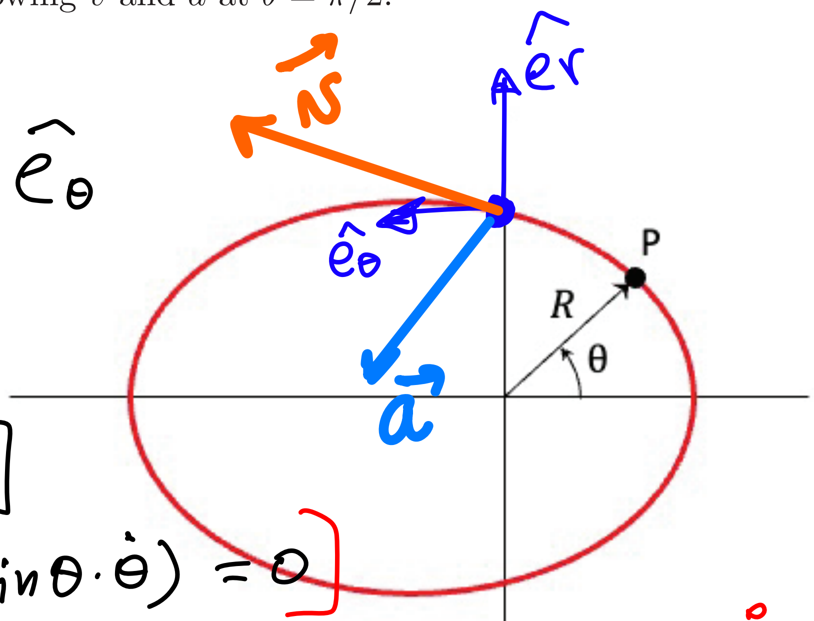
(b) Determine the velocity $\vec{v}$ and acceleration $\vec{a}$ vectors of P.

(c) Make a sketch showing $\vec{v}$ and $\vec{a}$ at $\theta = \pi/2$.

Solution

$$\vec{v} = \dot{R}\hat{e}_r + R\dot{\theta}\hat{e}_\theta$$

Need $\dot{R}$, $\dot{\theta}$



$R(2 + cos\theta) = b$

$$\frac{d}{dt}\left[R(2+\cos\theta) = b \right]$$

$$\Rightarrow \frac{d}{dt}\left(\dot{R}(2+\cos\theta) + R(-\sin\theta\cdot\dot{\theta}) = 0 \right]$$

$$\ddot{R}(2+\cos\theta) + \dot{R}(-\sin\theta\,\dot{\theta}) - \left[\dot{R}(\dot{\theta}\sin\theta) + R(\cancel{\ddot{\theta}}^{\,0}\sin\theta + \dot{\theta}^2\cos\theta) \right]$$

Use the following parameter in your work: $b = 2$ m and $\dot{\theta} = 3$ rad/s.

$$\Rightarrow \quad \dot{R} = \frac{R\,\cancel{\sin\theta}^{\,1}\,\dot{\theta}}{2+\cos\cancel{\theta}^{\,0}} \quad \Rightarrow \quad \dot{R} = \frac{R\dot{\theta}}{2}$$

$$\ddot{R}(2+\cos\theta) = -\dot{R}\sin\theta\,\dot{\theta} - \dot{R}\dot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$$

$$\ddot{R} = \frac{-2\dot{R}\dot{\theta}\cancel{\sin\theta}^{\,1} - \dot{\theta}^2\cancel{\cos\theta}^{\,0}}{2+\cancel{\cos\theta}^{\,0}} = -\dot{R}\dot{\theta}$$

$$\vec{v} = \frac{R\dot{\theta}}{2}\hat{e}_r + \frac{b\dot{\theta}}{2}\hat{e}_\theta$$

$$\vec{a} = \left(-\dot{R}\dot{\theta} - \frac{b\dot{\theta}^3}{2}\right)\hat{e}_r + \left(R\dot{\theta}^2\right)\hat{e}_\theta$$