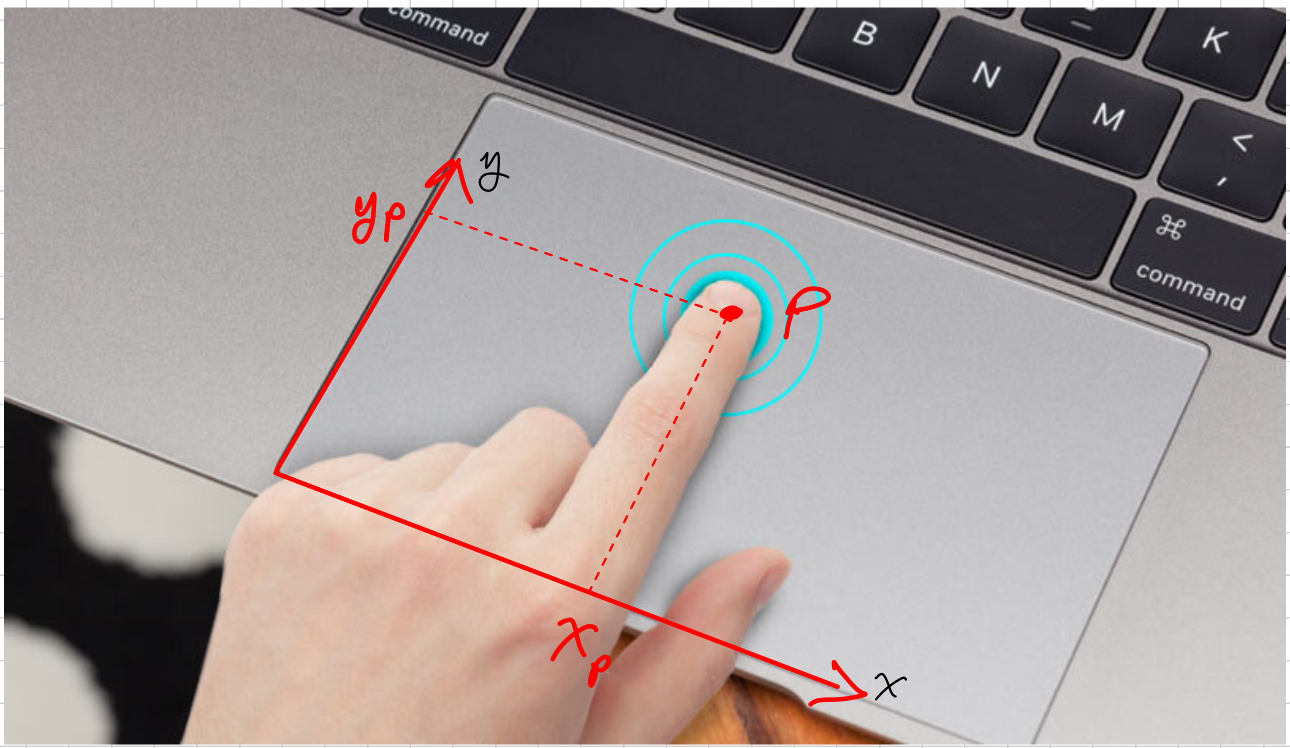
Cartesian Description:

$\hat{i}$ unit vector of $+x$ direction

$\hat{j}$ unit vector of $+y$ direction

Fixed!

Very common (and most intuitive) in math & physics



unit vector

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$x = x(t)$$

$$y = y(t)$$

$\hat{i}$ & $\hat{j}$ are NOT functions of t

$$x = a\sin t$$

So, finding the velocity and aacceleration of P in Cartesian description depends on our ability to take derivatives of x and y with respect to time. The challenging case is to deal with explicit or implicit time dependency.

→ If x and y are explicit functions of t, they can be directly differentiated in time.

→ If time dependency is hidden: $y = f(x)$, time is still present as $x = x(t)$. So y is an implicit function of time and x is an explicit function of time. Thus, here we need to use the chain rule of differentiation.

Example: $y = \sin x$ [m] Find $\vec{v}$, $\vec{a}$ when $x = \frac{\pi}{2}$

$\dot{x} = 3$ [m/s]

Solution:

$\dot{x} = 3$ m/s $\leftarrow$ constant $\Rightarrow \ddot{x} = 0$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \cos x \cdot \dot{x} = \dot{x} \cos x$$

$$\dot{y}\left(x = \frac{\pi}{2}\right) = 3 \cos\left(\frac{\pi}{2}\right) = 0 \text{ m/s}$$

$$\ddot{y} = \ddot{x} \cos x + (-\sin x \, \dot{x})(\dot{x}) = \ddot{x} \cos x - \dot{x}^2 \sin x$$

$$\ddot{y}\left(x = \frac{\pi}{2}\right) = 0 - 9 \sin\frac{\pi}{2} = -9 \text{ m/s}^2$$

$\therefore$

$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} = 3\,\hat{i} + 0\,\hat{j} \quad [\text{m/s}]$$

$$\vec{a} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} = 0\,\hat{i} - 9\,\hat{j} \quad [\text{m/s}^2]$$

Example 1.A.1

Given: Pin P is constrained to move along a elliptical ring whose shape is given by $x^2/a^2+y^2/b^2=1$ (where x and y are given in mm). The pin is also constrained to move within a horizontal slot that is moving upward at a constant speed of v.

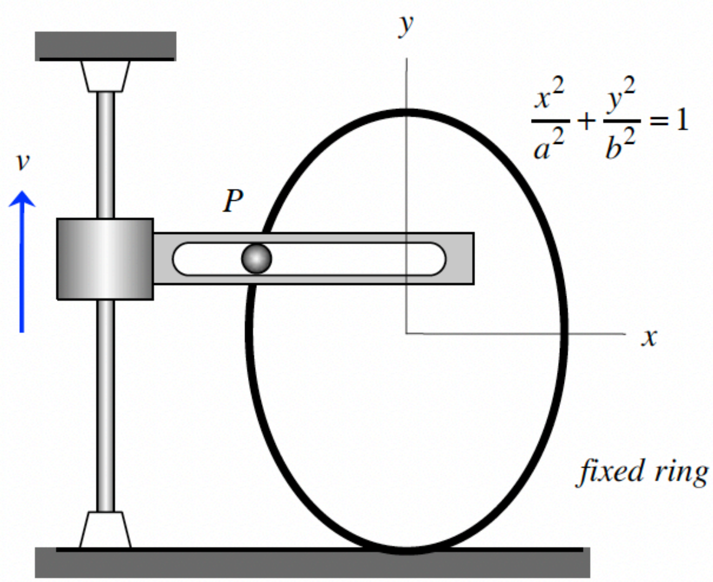
Find: Determine:
(a) The velocity of pin P at the position corresponding to $y = 6$ mm; and
(b) The acceleration of pin P at the position corresponding to $y = 6$ mm.

Use the following parameters in your analysis: $a = 5$ mm, $b = 10$ mm, $v = 30$ mm/s.

Solution

$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$; $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$\dot{y}$ = constant $\Rightarrow \ddot{y} = 0$



$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$x = -a\sqrt{1-\frac{y^2}{b^2}}$$

Example 1.A.1

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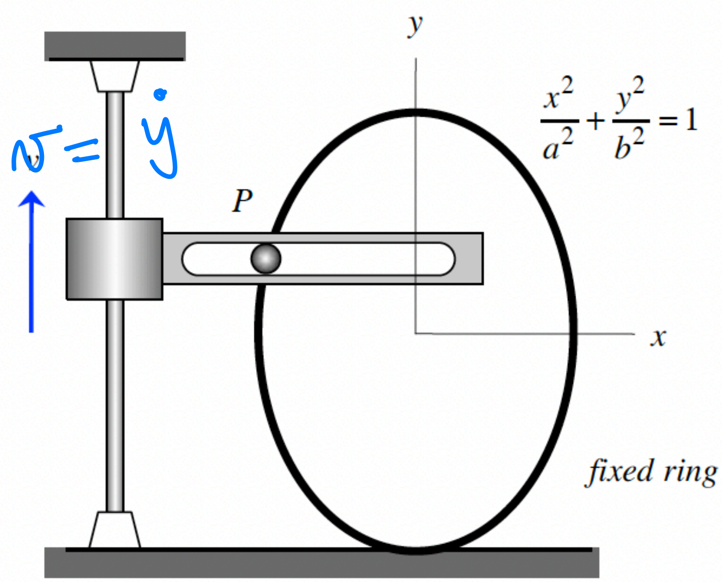
Find: Determine:
(a) The velocity of pin P at the position corresponding to $y = 6$ mm; and
(b) The acceleration of pin P at the position corresponding to $y = 6$ mm.

y: explicit
x: implicit

Use the following parameters in your analysis: $a = 5$ mm, $b = 10$ mm, $v = 30$ mm/s.

Solution: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

$$\vec{a} = \frac{d}{dt}[\vec{v}] = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$



Differentiate directly:

$$\frac{d}{dt}\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right]$$

$$\frac{1}{a^2}(\cancel{2}x\dot{x}) + \frac{1}{b^2}(\cancel{2}y\dot{y}) = 0$$

$\dot{y}$ given; solve for $\dot{x}$

$$\frac{1}{a^2}(x\dot{x}) = -\frac{1}{b^2}(y\dot{y}) \Rightarrow \dot{x} = -\frac{a^2 y}{b^2 x}\dot{y}$$

Substitute $\dot{x}$ and $\dot{y}$ in $\vec{v}$

$$\vec{v} = -\frac{a^2 y}{b^2 x}\dot{y}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$$ ← plug in numeric values here

$$\vec{v} = -11.25\,\hat{\imath} + 30\,\hat{\jmath} \quad [m/s]$$

Similarly for acceleration:

$$\frac{d}{dt}\left(\frac{1}{a^2}x\dot{x} + \frac{1}{b^2}y\dot{y} = 0\right) \Rightarrow \frac{1}{a^2}(\dot{x}\dot{x} + x\ddot{x}) + \frac{1}{b^2}(\dot{y}\dot{y} + y\overset{0}{\cancel{\ddot{y}}}) = 0$$

$\ddot{y}$ given, solve for $\ddot{x}$

$$\frac{1}{a^2}(\dot{x}^2 + x\ddot{x}) + \frac{1}{b^2}\dot{y}^2 = 0 \Rightarrow \ddot{x} = -\frac{a^2\dot{y}^2}{b^2 x} - \frac{\dot{x}^2}{x}$$

Substitute $\ddot{x}$ and $\ddot{y}$ in $\vec{a}$ $(\ddot{y} = 0)$

$$\vec{a} = \left[-\frac{a^2\dot{y}^2}{b^2 x} - \frac{\dot{x}}{x}\right]\hat{\imath}$$ ← plug numeric values here

$$\vec{a} = -87.89\,\hat{\imath} \quad [m/s^2]$$

Example 1.A.2

Given: A particle P moves on a path whose Cartesian components are given by the following functions of time (where both components are given in inches and time t is given in seconds):

$$x(t) = t^3 + 10$$
$$y(t) = 2\cos 4t$$

Find: Determine at the time $t = 2$ s:
(a) The velocity vector of P;
(b) The acceleration of P; and
(c) The angle between the velocity and acceleration vectors of P.

Solution:

$$\dot{x}(t) = 3t^2 \Rightarrow \ddot{x}(t) = 6t$$

$$\dot{y}(t) = 2(-\sin 4t)(4) \Rightarrow \ddot{y}(t) = -8\cos 4t\,(4)$$

Plug into $\vec{v}$ and $\vec{a}$

$$\vec{v} = 3t^2\,\hat{\imath} - 8\sin 4t\,\hat{\jmath}$$

$$\vec{a} = 6t\,\hat{\imath} - 32\cos 4t\,\hat{\jmath}$$

Example 1.A.2

Given: A particle P moves on a path whose Cartesian components are given by the following functions of time (where both components are given in inches and time t is given in seconds):

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Find: Determine at the time $t = 2$ s:

(a) The velocity vector of P;
(b) The acceleration of P; and
(c) The angle between the velocity and acceleration vectors of P.

To fin the angle, use the dot product.

$$\vec{v} \cdot \vec{a} = |\vec{v}||\vec{a}| \cos\theta$$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{a}}{|\vec{v}||\vec{a}|}\right)}$$, where $|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}$

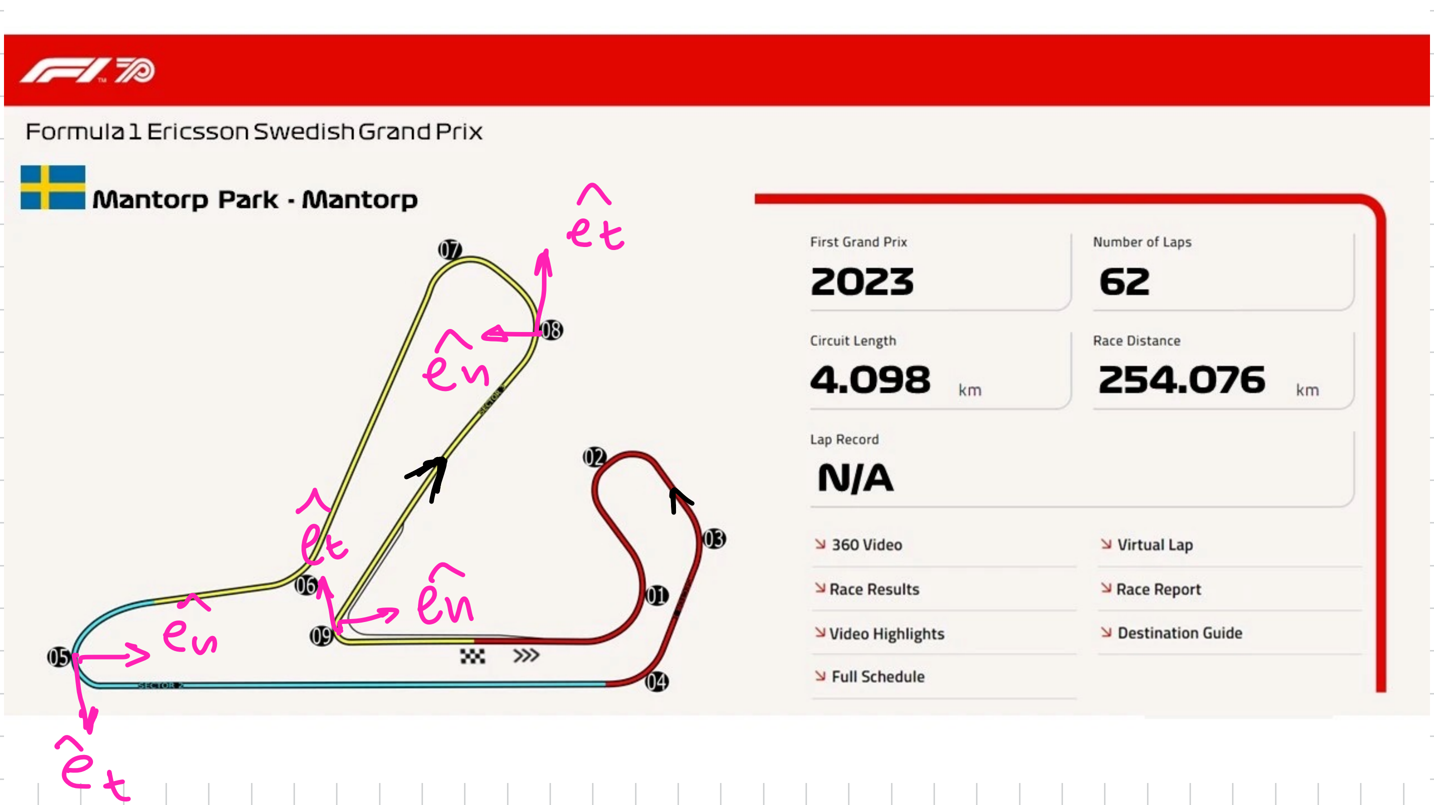
and $|\vec{a}| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

PATH KINEMATICS

$\hat{e}_t$ unit vector tangent to moving trajectory

$\hat{e}_n$ unit vector perpendicular to $\hat{e}_t$ pointing toward the turning center of curvature (ALWAYS)

Example: a racetrack!

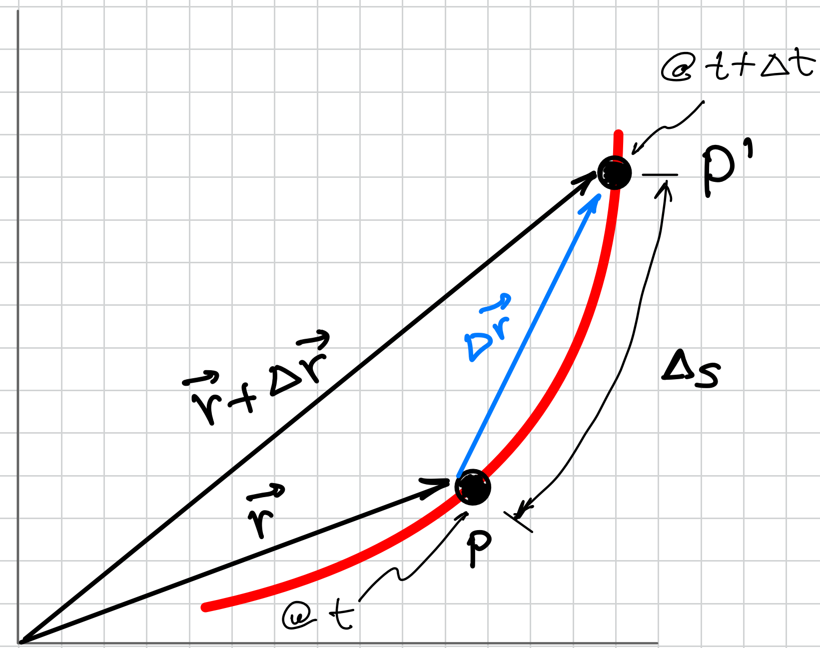


For path description:

The position of P is referenced by the arc length distance s, measured along the path

The path geometry is known $\longrightarrow$ $\vec{r} = \vec{r}\,[s(t)]$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt}$$



What does $\dfrac{ds}{dt}$ mean as s is not a vector?

↳ The "SPEED" of the particle

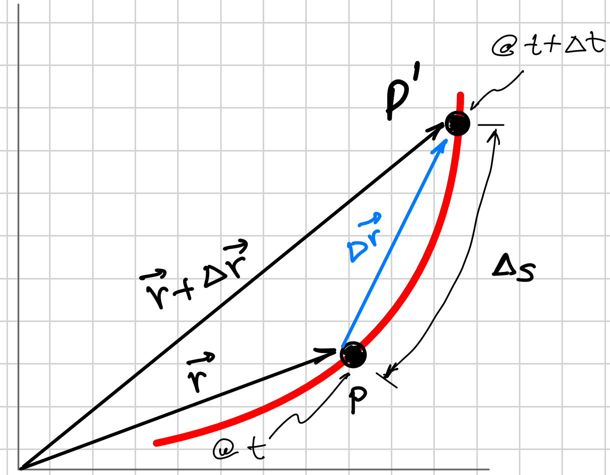
$\longrightarrow$ |speed| ≠ velocity

speed ↓ scalar ($|\vec{v}|$)

velocity ↓ vector (has direction)

Definition of path description:

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \vec{r}}{\Delta s}$$

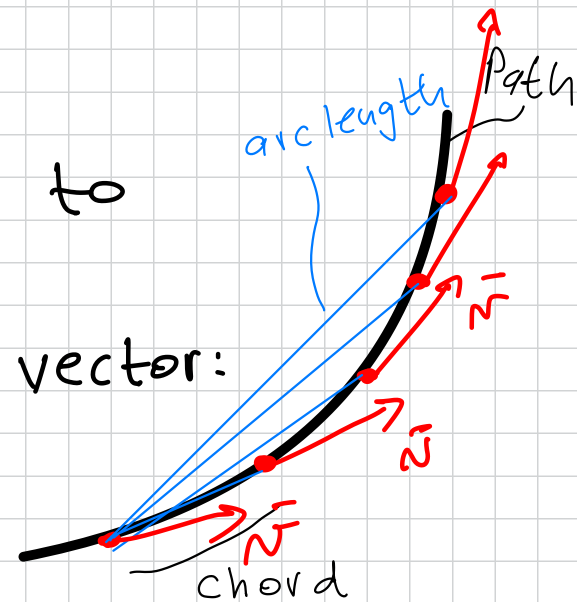


The smaller Δs gets, $|\Delta r| \simeq \Delta s$

Thus, $\frac{d\vec{r}}{ds}$ becomes unit vector ($\hat{e}_t$)

(magnitude 1)

Also, as $\Delta s \to 0$, $\Delta\vec{r}$ becomes tangent to the path of P. This is actuall the direction of the velocity vector:



$$\vec{v} = v\,\hat{e}_t$$

Does $\hat{e}_t$'s direction change over time?

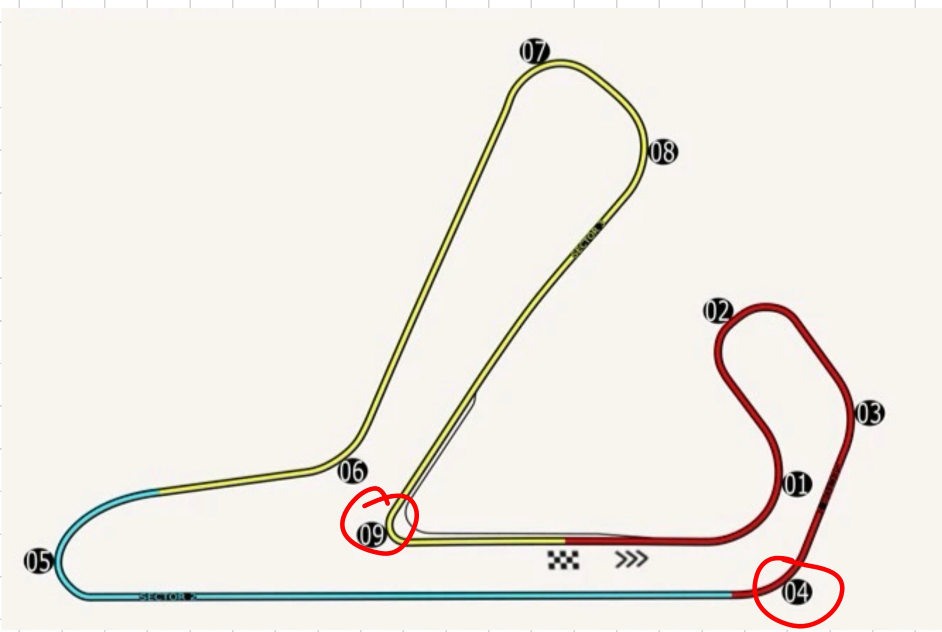
YES! both $\hat{e}_n$ and $\hat{e}_t$ do

If we take the derivative of $\vec{v}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\,\hat{e}_t) = \overbrace{\frac{dv}{dt}}^{\dot{v}}\,\hat{e}_t + v\,\frac{d}{dt}(\hat{e}_t)$$

Note that the unit vector $\hat{e}_t$ is a function of position, not time. Since we want to find the dependence of position with time, we use the chain rule:

$$\frac{d}{dt}(v\,\hat{e}_t) = \dot{v}\,\hat{e}_t + \underbrace{v\,\frac{d\hat{e}_t}{ds}}\cdot\underbrace{\frac{ds}{dt}}$$



How road curves @ my location e.g.: 09 and 04 curves are quite different!

How fast I'm driving

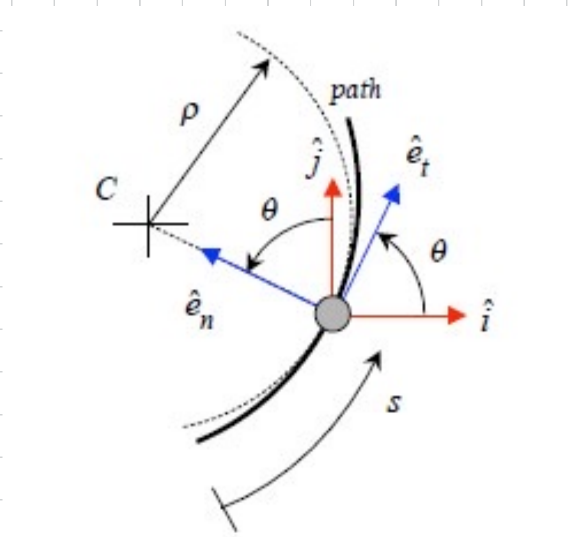
However, $\dfrac{d\hat{e}_t}{ds} = \dfrac{d\hat{e}_t}{d\theta} \cdot \dfrac{d\theta}{ds} \Rightarrow \vec{a} = \dot{N}\hat{e}_t + N^2 \dfrac{d\hat{e}_t}{ds}$

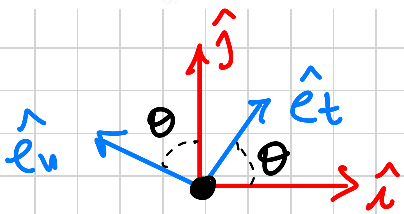
$\dfrac{d\hat{e}_t}{d\theta} \cdot \dfrac{d\theta}{ds}$

This is a consequence of $\hat{e}_t$ being a function of both angle wrt $\hat{\imath}$-$\hat{\jmath}$ (θ), and also s.

Let's look @ what happens when P turns right or left $\Rightarrow$ The unit vectors change accordingly...

PATH TURNING RIGHT



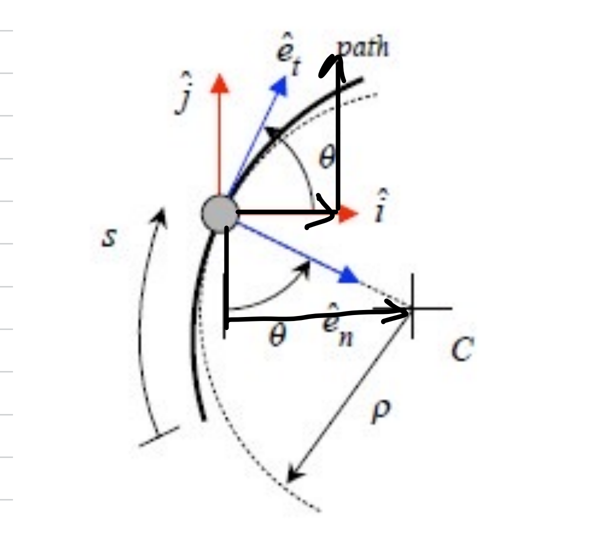


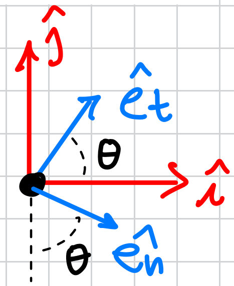
$$\hat{e}_n = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\hat{e}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\frac{d\hat{e}_t}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{e}_n$$

PATH TURNING LEFT



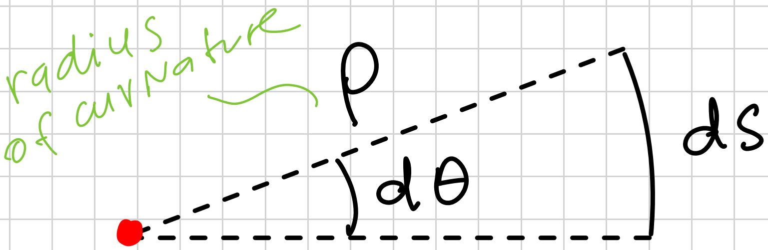


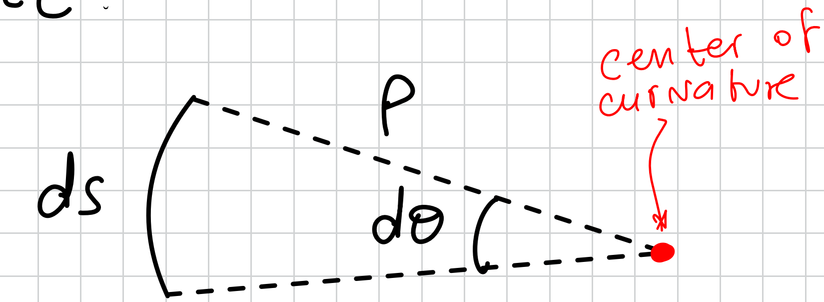
$$\hat{e}_n = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$$

$$\hat{e}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\frac{d}{d\theta}\hat{e}_t = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = -\hat{e}_n$$

Now let's connect the dependence of angle θ with that of distance:





Left turn

$$ds = \rho\, d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\therefore \quad \frac{d\hat{e}_t}{d\theta}\cdot\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

Right turn

$$ds = -\rho\, d\theta \Rightarrow \frac{d\theta}{ds} = -\frac{1}{\rho}$$

$$\therefore \quad \frac{d\hat{e}_t}{d\theta}\cdot\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

Curvature is the rate @ tangent direction changes wrt distance traveled. When turning left, tangent direction grows. When turning right, tangent direction shrinks (e.g. steering wheel)

So, in general

$$\frac{d\hat{e}_t}{d\theta} \cdot \frac{d\theta}{ds} = \frac{1}{\rho} \hat{e}_n$$

$\rho \rightarrow$ radius of curvature.

Altogether, all these results produce the following relationship:

$$\boxed{\vec{a} = \underbrace{\dot{v}\,\hat{e}_t}_{} + \underbrace{\frac{v^2}{\rho}\,\hat{e}_n}_{}}$$

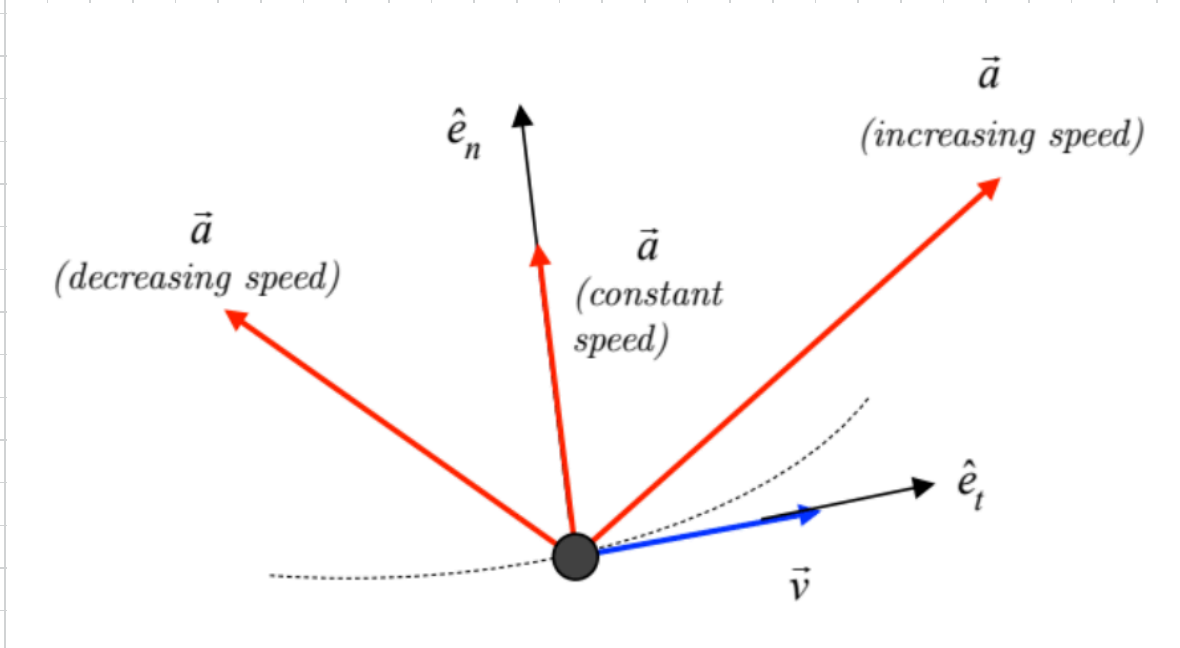
- Comes from speeding/slowing down (e.g. gas pedal)
- Always tangent to path
- $\dot{v}$ is the rate of change of the speed of P

- Comes from changing direction (e.g. Steering wheel)
- Normal to center of curvature
- Centripetal acceleration (ALWAYS)

$\dot{v} > 0 \Rightarrow$ speed $\uparrow \Rightarrow a_t$ (+) ($\vec{a}$ points Forward of $\hat{e}_n$)

$\dot{v} = 0 \Rightarrow$ speed constant $\Rightarrow a_t = 0$ ($\vec{a}$ aligned w/ $\hat{e}_n$)

$\dot{v} < 0 \Rightarrow$ speed $\downarrow \Rightarrow a_t$ (−) ($\vec{a}$ points backward)



$$\dot{v} \neq |\vec{a}|$$

Rate of change of speed = How fast speed changes

Magnitude of acceleration vector = accounts for both

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}$$

like any vector.

Example 1.A.3

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

Find: The radius of curvature of the path of the jet at A.

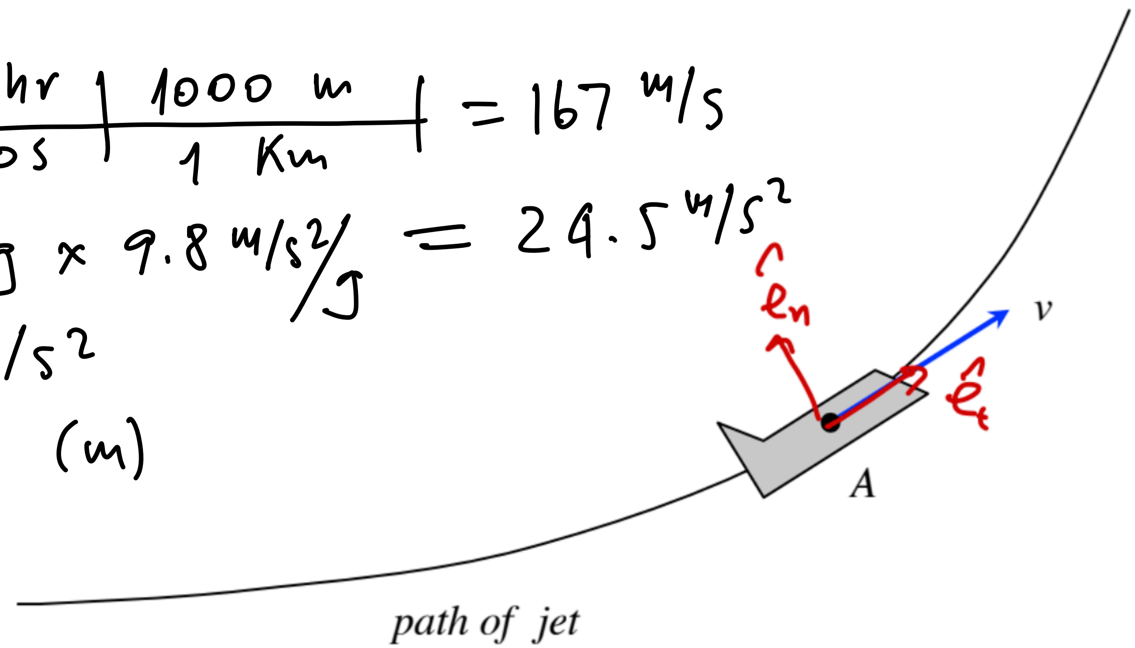
Solution :

$$600 \, \frac{\text{km}}{\text{hr}} \left| \frac{1 \text{ hr}}{3600 \text{ s}} \right| \frac{1000 \text{ m}}{1 \text{ km}} \Bigg| = 167 \text{ m/s}$$

$$|\bar{a}| = 2.5 \cancel{g} \times 9.8 \, \frac{\text{m/s}^2}{\cancel{g}} = 24.5 \text{ m/s}^2$$

$$a_t = 5 \text{ m/s}^2$$

$$\rho = ? \quad (\text{m})$$



$$\vec{a} = a_t \hat{e}_t + a_n \hat{e}_n = \dot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n$$

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2} \Rightarrow |\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2$$

$$\frac{v^2}{\rho} = \sqrt{|\bar{a}|^2 - \dot{v}^2} \Rightarrow \rho = \frac{v^2}{\sqrt{|\bar{a}|^2 - \dot{v}^2}}$$

$$\rho = \frac{167^2}{\sqrt{25^2 - 5^2}} = 1163 \quad [m]$$