ME 274: Basic Mechanics II

Lecture 5: Planar kinematics: rigid bodies

PURDUE UNIVERSITY | School of Mechanical Engineering

Announcements

Homework changes due to snow cancelations:

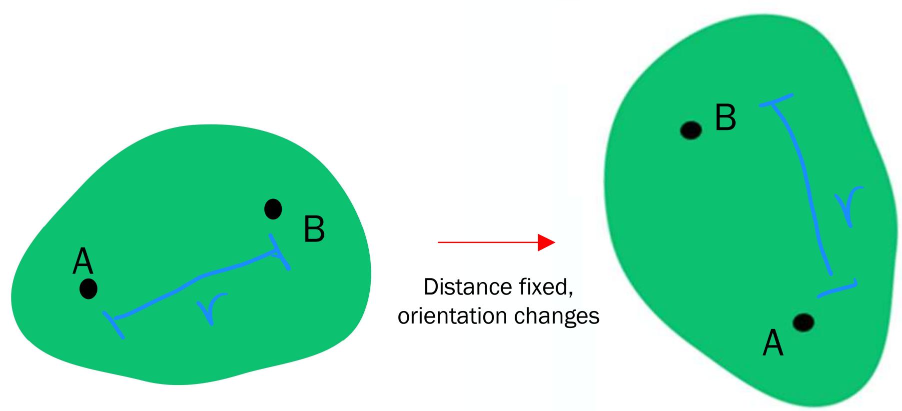
HW 2.B, 2.C, 2.D due on Friday

HW 2.A canceled

HW 1.I and 1.J still due tonight

What are rigid bodies?

Any object for which the distance between any two points on the object remains fixed regardless of the motion of the object



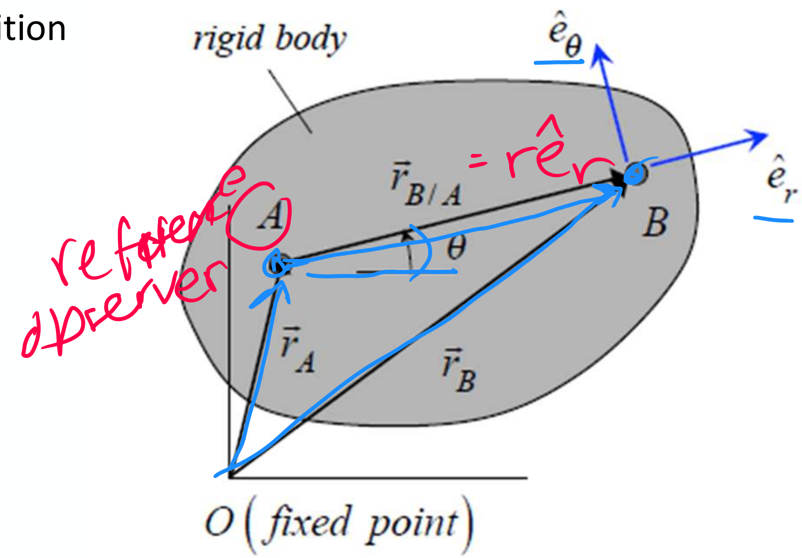
Point kinematics→ we treat the object as a particle $\rightarrow \vec{r}, \vec{v}, \vec{a}$
Rigid body kinematics →we care about the **position, geometry, and orientation** of the object

If we know the **motion of one point** and the **rotation of the body**, we can determine the **motion of every point** on the body

Kinematic Equations for Planar Rigid Body Motion

- Position of point B with respect to point A found using relative position expression

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$



Using the polar description of $\vec{r}_{B/A}$:

- Fixed points → $|\vec{r}_{B/A}| = r = const.$, $\dot{r}_{B/A} = 0$
- Changing θ → $\vec{r}_{B/A} \neq const.$

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A = r\hat{e}_r$$

Take time derivatives for velocity and acceleration:

$$\vec{V}_{B/A} = \vec{V}_B - \vec{V}_A = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_t$$

Deriving the Rigid Body Velocity Equation

We can derive a **description agnostic expression** by writing in terms of **vector operations**

$\vec{v}_{B/A} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \vec{v}_B - \vec{v}_A$ ← specific to polar coordinates

we know $|\vec{r}_{B/A}| = r = \text{const}$

$\dot{r} = 0$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \quad \leftarrow \text{relative}$$

$$= \cancel{\dot{r}\hat{e}_r} + r\dot{\theta}\hat{e}_\theta \quad \leftarrow \text{polar}$$

$$\vec{V}_{B/A} = r\dot{\theta}\hat{e}_\theta$$

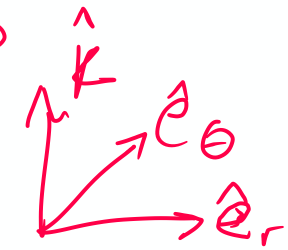
$$\rightarrow = r\dot{\theta}\left(\hat{k} \times \hat{e}_r\right)$$

scalar

$$= \left(\dot{\theta}\hat{k}\right) \times (r\hat{e}_r)$$

$$\vec{V}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$$

$\hat{e}_\theta = \hat{k} \times \hat{e}_r$



$r\hat{e}_r \Rightarrow$ relative position vector $\vec{r}_{B/A}$

$\dot{\theta}\hat{k} = \vec{\omega}$

where $\vec{\omega} = \dot{\theta}\hat{k}$ is the "angular velocity" of the rigid body.

Deriving the Rigid Body Acceleration Equation

We can derive a **description agnostic expression** by writing in terms of **vector operations**

$$\vec{a}_{B/A} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = \vec{a}_B - \vec{a}_A$$ ← specific to polar coordinates

$|\vec{r}_{B/A}| = r = const \rightarrow \dot{r} = 0 = \ddot{r}$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$ ← relative

$$= \left(\cancel{\ddot{r}}^{\,0} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}}^{\,0}\right)\hat{e}_\theta$$ ← polar

$$\vec{a}_{B/A} = \left(-r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta}\right)\hat{e}_\theta$$ → in terms of $\hat{k}$

$$= \left(r\dot{\theta}^2\right)\left[\hat{k} \times \left(\hat{k} \times \hat{e}_r\right)\right] + \left(r\ddot{\theta}\right)\left(\hat{k} \times \hat{e}_r\right)$$

$$= \dot{\theta}\hat{k} \times \left[\left(\dot{\theta}\hat{k}\right) \times \left(r\hat{e}_r\right)\right] + \left(\ddot{\theta}\hat{k}\right) \times \left(r\hat{e}_r\right)$$

$$= \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

$$\left.\begin{aligned} \hat{e}_\theta &= \hat{k} \times \hat{e}_r \\ \hat{e}_r &= -\hat{k} \times (\hat{k} \times \hat{e}_r) \end{aligned}\right\}$$

where $\vec{\alpha} = \theta k$ is the "angular acceleration" of the rigid body.

To find the velocity/acceleration of point B

→

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

transtration rotation

$$\vec{a}_{B/A} = \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{-\omega^2 \vec{r}_{B/A}} + \vec{\alpha} \times \vec{r}_{B/A} \leftarrow 3D$$

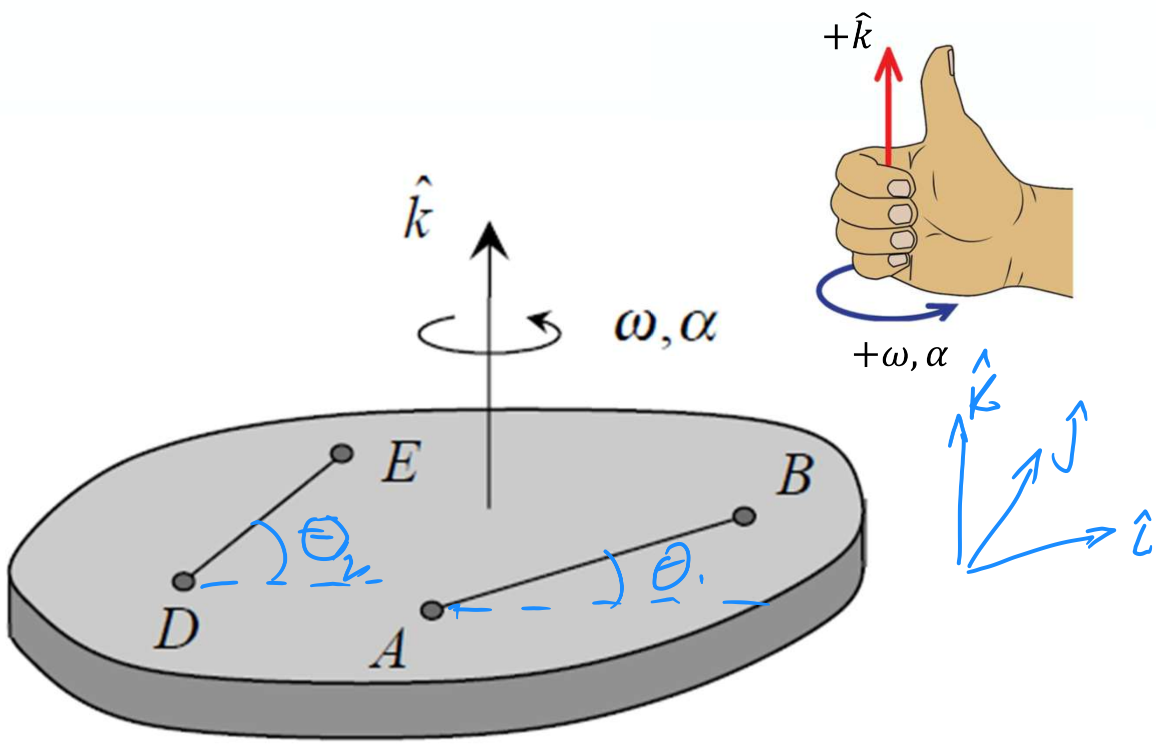
$$2D \quad \vec{a}_{B/A} = -\omega^2 \vec{r}_{B/A} + \vec{\alpha} \times \vec{r}_{B/A}$$

Angular Velocity and Acceleration

Angular velocity: $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$

Angular acceleration: $\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{\mathrm{k}}$

Describe the motion of the **BODY** and is the same for any set of points – i.e. a rigid body will only have one angular velocity and acceleration



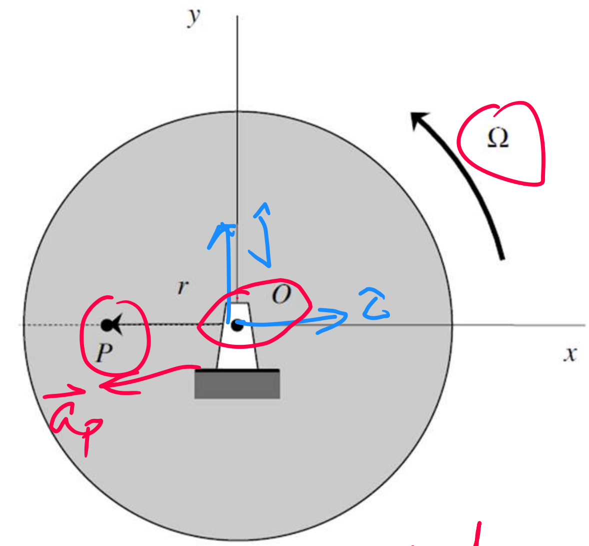
The direction $\hat{k}$ denotes the axis the body rotates around. The sign denotes the direction (ex. Clockwise, ccw)

$$\theta_2 = \theta_1 + \text{const}$$

$$\dot{\theta}_2 = \dot{\theta}_1 + 0 = \omega$$

$$\ddot{\theta}_2 = \ddot{\theta}_1 = \alpha$$

Example 2.A.1



Given: The disk shown is rotating at a non-constant rate of Ω about a fixed axis passing through its center O. At a particular instant, the acceleration vector of point P on the disk is $\vec{a}_P$.

Find: Determine:

(a) The angular velocity of the disk at this instant; and

(b) The angular acceleration of the disk at this instant.

Use the following parameters in your analysis: $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2 and $r = 0.4$ m. Also, be sure to write your answers as vectors.

$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} - \omega^2 \vec{r}_{P/O}$$

$$a_{px}\hat{\imath} + a_{py}\hat{\jmath} = 0 + \alpha\hat{k} \times -r\hat{\imath} - \omega^2(-r\hat{\imath})$$

$$= -\alpha r\hat{\jmath} + \omega^2 r\hat{\imath}$$

$O \rightarrow$ pinned

$\vec{v}_O = 0 \quad \vec{a}_O = 0$

$$\hat{\imath}: \quad a_{px} = \omega^2 r \rightarrow \omega = \pm\sqrt{\frac{a_{px}}{r}} = \pm\sqrt{\frac{3}{0.4}} = \pm 2.74 \frac{rad}{s}$$

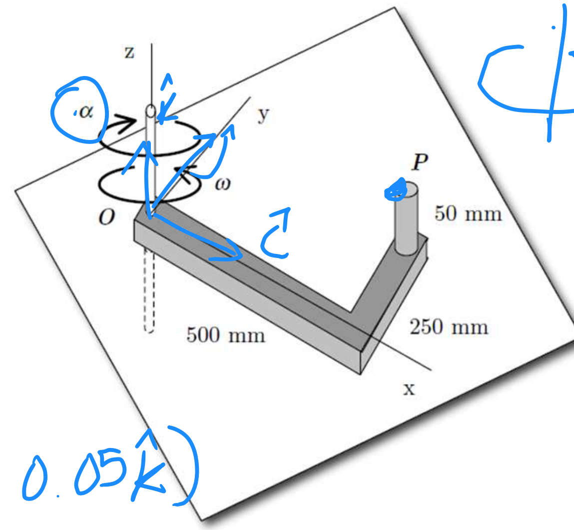
$$\vec{\omega} = \pm 2.74 \frac{rad}{s}\hat{k}$$

$$\hat{\jmath}: \quad a_{py} = -\alpha r \rightarrow \alpha = -\frac{a_{py}}{r} = -\frac{4}{0.4} = -10 \text{ rad/s}^2$$

$$\vec{\alpha} = -10\hat{k} \text{ rad/s}^2$$

Example 2.A.3

Given: The system shown below rotates about a vertical shaft at point O, such that $\omega = 2$ rad/s and $\alpha = 3$ rad/s^2.



Find: Determine:
(a) The velocity of point P; and
(b) The acceleration of point P.

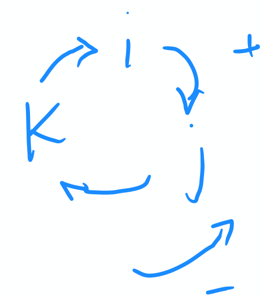
O pinned → $V_o = 0$
$a_o = 0$

$$\vec{V}_P = \cancel{\vec{V}_o}^{\,0} + \vec{\omega}\times\vec{r}_{P/O}$$

$$= \vec{\omega}\times\vec{r}_{P/O} = \omega\hat{k}\times(0.5\hat{i} + .25\hat{j} + 0.05\hat{k})$$

$$\vec{V}_P = 0.5\omega\hat{j} - .25\omega\hat{i}$$

$$0.5(2)\hat{j} - 0.25(2)\hat{i} = \boxed{1\hat{j} - 0.5\hat{i} \ \text{m/s}}$$



$$\vec{a}_P = \cancel{\vec{a}_o}^{\,0} + \alpha\times\vec{r}_{P/O} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O})$$

$$= \alpha\hat{k}\times(0.5\hat{i} + .25\hat{j} + 0.05\hat{k})$$

$$+ \omega\hat{k}\times[\omega\hat{k}\times(0.5\hat{i} + 0.25\hat{j} + 0.05\hat{k})]$$

$$= 0.5\alpha\hat{j} - .25\alpha\hat{i} + \omega\hat{k}\times(0.5\omega\hat{j} - 0.25\omega\hat{i})$$

$$\rightarrow 0.5\omega^2\hat{i} - 0.25\omega^2\hat{j}$$

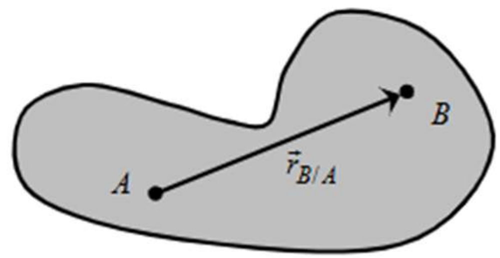
$$\vec{a}_P = (-.25\alpha - .5\omega^2)\hat{i} + (.5\alpha - .25\omega^2)\hat{j}$$

→ plug in + solve

Summary: Rigid Body Kinematics 1

PROBLEM: Two points A and B on the same rigid body undergoing planar motion.

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A} \right)$$



COMMENTS:

- $\vec{\omega}$ and $\vec{\alpha}$ are the ***angular velocity*** and ***angular acceleration*** vectors of the body. These are the same for ANY two points A and B.

- $\vec{r}_{B/A}$ points ***FROM*** point A ***TO*** point B.

- If A and B lie in the same plane, then: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$

- From where did these equations come? From the general motion of two points (Chapter 1) with the constraint that $\left| \vec{r}_{B/A} \right|$.