ME 274: Basic Mechanics II

Lecture 4: Point Kinematics, Joint Description

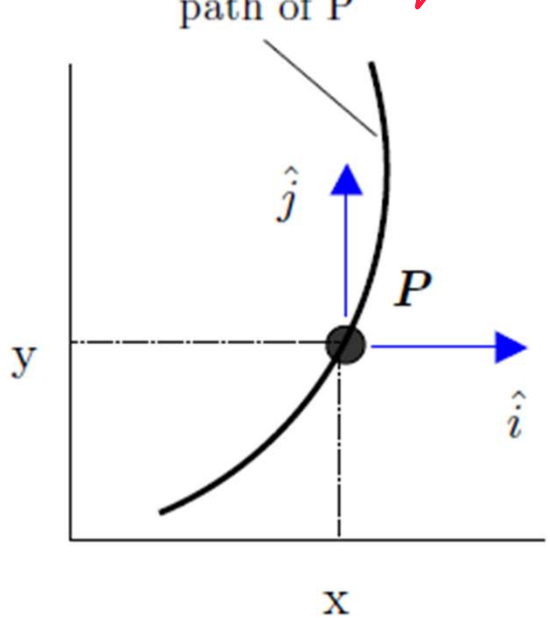
PURDUE UNIVERSITY | School of Mechanical Engineering

Kinematic Descriptions

Cartesian description:

- Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath}$
- Fixed direction basis vectors $\hat{\imath}, \hat{\jmath}$
- $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$
- $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

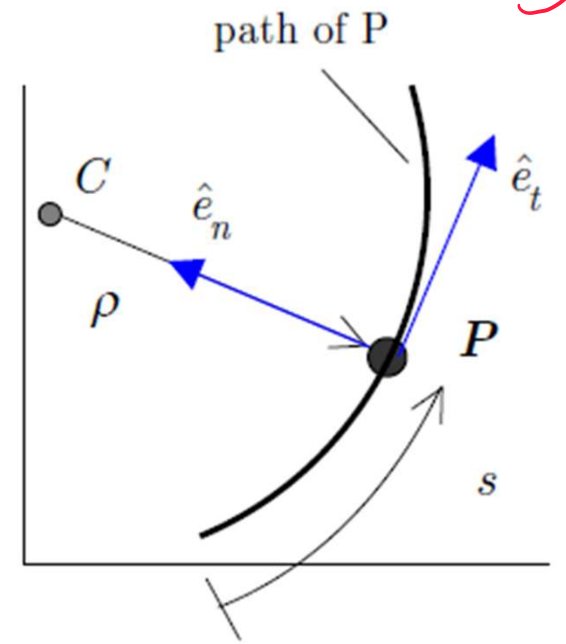
good for horizontal + vertical motion



Path description:

- Distance along path: $s(t)$
- $\hat{e}_t, \hat{e}_n$ depend on path geometry ← move + rotate
- $\vec{v} = v\hat{e}_t$
- $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

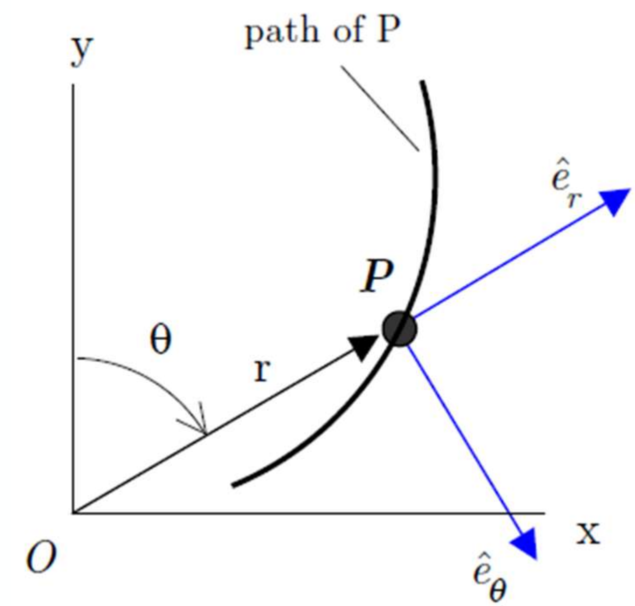
good for speed/turning



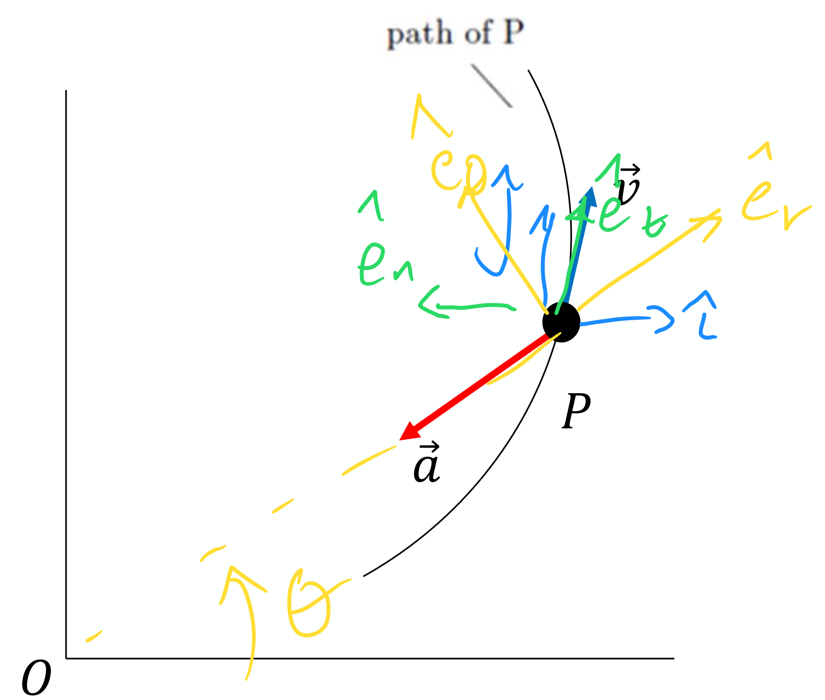
Polar description:

- Position: $\vec{r} = r\hat{e}_r$ θ
- $\hat{e}_r, \hat{e}_\theta$ change with particle motion
- $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
- $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

good for central observer



Different Descriptions – Same Motion



Velocity: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v\,\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{v} = \vec{v} = \vec{v} = \vec{v}$

Acceleration: $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

$\vec{a} = \vec{a} = \vec{a}$

convert between descriptions

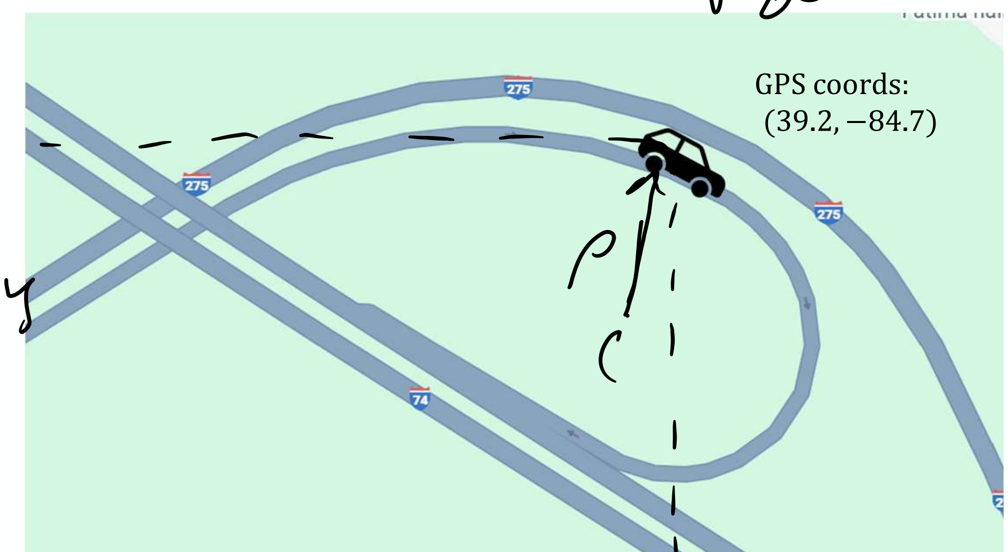
Joint Description: Combined Usage of Kinematic Descriptions

Example: Car moving around a turn

The motion of a car is described in Cartesian coordinates

Questions we care about:

- Is the car speeding up or slowing down?
- How hard is the car turning?



Example: Cheering for a runner

Tracking app gives path information:

- Distance along course
- Current speed and split times

Questions we care about:

- Where to stand to cheer – runner's absolute position at a time

Converting between descriptions: Projection

MOTIVATING EXAMPLE:
Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$

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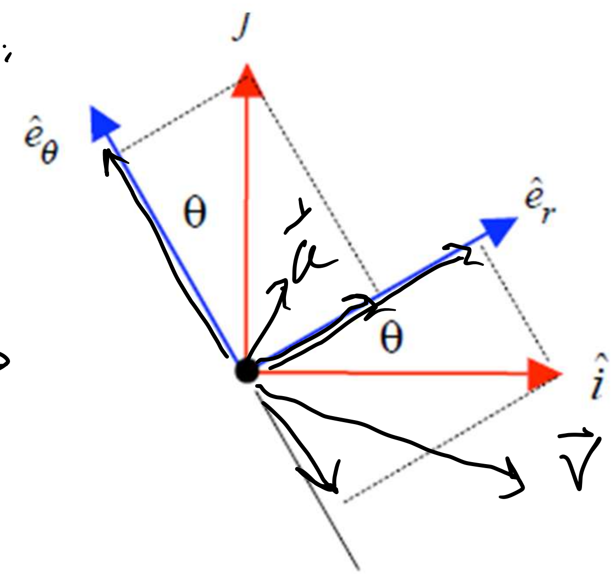


Solution:

Write **target** basis vectors **in terms of given** basis vectors

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = 0.8\hat{e}_r - 0.6\hat{e}_\theta$$
$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = 0.6\hat{e}_r + 0.8\hat{e}_\theta$$

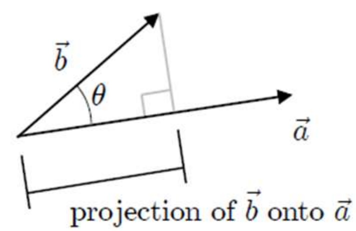
Calculate velocity components by finding projection of $\vec{v}$ onto $\hat{\imath}, \hat{\jmath}$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} \qquad \vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

$$\dot{x} = \vec{v}\cdot\hat{i} = (10\hat{e}_r - 20\hat{e}_\theta)\cdot(0.8\hat{e}_r - 0.6\hat{e}_\theta)$$
$$\ddot{x} = \vec{a}\cdot\hat{i} \qquad = 8 + 12 = 20\text{ m/s}$$
$$\dot{y} = \vec{v}\cdot\vec{j} = (3e_r + 2e_\theta)\cdot(0.6\hat{e}_r + 0.8\hat{e}_\theta)$$
$$\ddot{y} = \vec{a}\cdot\vec{j} \qquad = 1.8 + 1.6 = 3.4\text{ m/s}$$

$$\vec{V} = 20\hat{i} + 3.4\hat{j}$$

Projection of $\vec{b}$ onto $\vec{a} = |\vec{b}||\hat{e}_a|\cos\theta = \vec{b} \bullet \hat{e}_a$



Component extraction via projection

Cartesian:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

$$\dot{x} = \vec{v} \bullet \hat{i} \qquad \dot{y} = \vec{v} \bullet \hat{j}$$
$$\ddot{x} = \vec{a} \bullet \hat{i} \qquad \ddot{y} = \vec{a} \bullet \hat{j}$$

Polar:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\dot{r} = \vec{v} \bullet \hat{e}_r \qquad r\dot{\theta} = \vec{v} \bullet \hat{e}_\theta$$
$$\ddot{r} - r\dot{\theta}^2 = \vec{a} \bullet \hat{e}_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \vec{a} \bullet \hat{e}_\theta$$

1) target basis in terms of given
2) take projection

Converting between descriptions: Coefficient Balancing

MOTIVATING EXAMPLE:

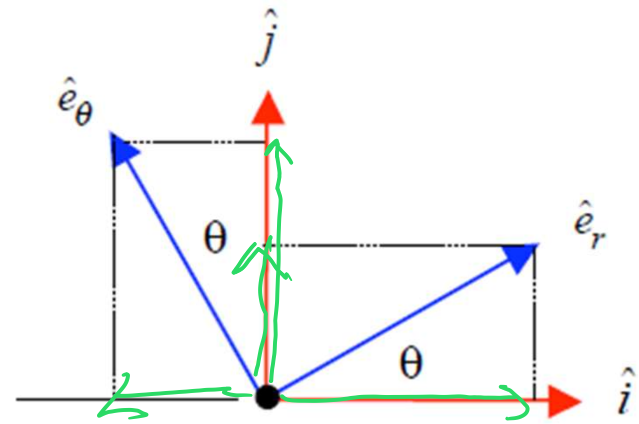
Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$.

Solution:

Write **given** basis vectors **in terms of target** basis vectors

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = 0.8\,\hat{i} + 0.6\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = -0.6\,\hat{i} + 0.8\,\hat{j}$$



Directly substitute into expression for $\vec{v}$:

$$\vec{V} = (10\hat{e}_r - 20\hat{e}_\theta) = 10(0.8\hat{i} + 0.6\hat{j}) - 20(-0.6\hat{i} + 0.8\hat{j})$$

$$= 8\hat{i} + 6\hat{j} + 12\hat{i} - 16\hat{j} = 20\hat{i} - 10\hat{j} \text{ m/s}$$

$$\vec{a}\ (3\hat{e}_r + 2\hat{e}_\theta) = 3(0.8\hat{i} + 0.6\hat{j}) + 2(-0.6\hat{i} + 0.8\hat{j})$$

$$= 1.2\hat{i} + 3.4\hat{j} \text{ m/s}^2$$

Example 2

MOTIVATING EXAMPLE:

Suppose the velocity and acceleration of a particle are known in terms of their Cartesian components as: $\vec{v} = (30\hat{i} - 40\hat{j})$ m/s and $\vec{a} = (-10\hat{j})$ m/s^2. From this, we want to find the speed v, rate of change of speed $\dot{v}$ and the radius of curvature ρ of the path of the particle (path description variables).

$$v = |\vec{v}| = \sqrt{30^2 + (-40)^2} = \boxed{50 \text{ m/s}}$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$\vec{v} = v\hat{e}_t \rightarrow \hat{e}_t = \frac{\vec{v}}{v} = \frac{30\hat{i} - 40\hat{j}}{50} = 0.6\hat{i} - 0.8\hat{j}$$

$$\dot{v} = \vec{a} \cdot \hat{e}_t = (-10\hat{j}) \cdot (0.6\hat{i} - 0.8\hat{j}) = 8 \text{ m/s}^2$$

$$|a|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \rightarrow \left(\frac{v^2}{\rho}\right)^2 = |a|^2 - \dot{v}^2$$

$$\rightarrow \rho = \frac{v^2}{\sqrt{|a|^2 - \dot{v}^2}} = \frac{50^2}{\sqrt{10^2 - 8^2}}$$

$$\boxed{= \frac{2500}{6} \text{ m}}$$

Example 1.C.2

Given: Pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant shown, guide A moves to the right with a speed of v_A, a speed that is changing at a rate of $\dot{v}_A$. At the same time, B is moving downward with a speed of v_B with a rate of change of speed of $\dot{v}_B$.

↳ given cartesian

Find:

(a) The rate of change of speed ($\dot{V}$) of P at this instant; and

(b) The radius of curvature ρ of the path followed by P at this instant.

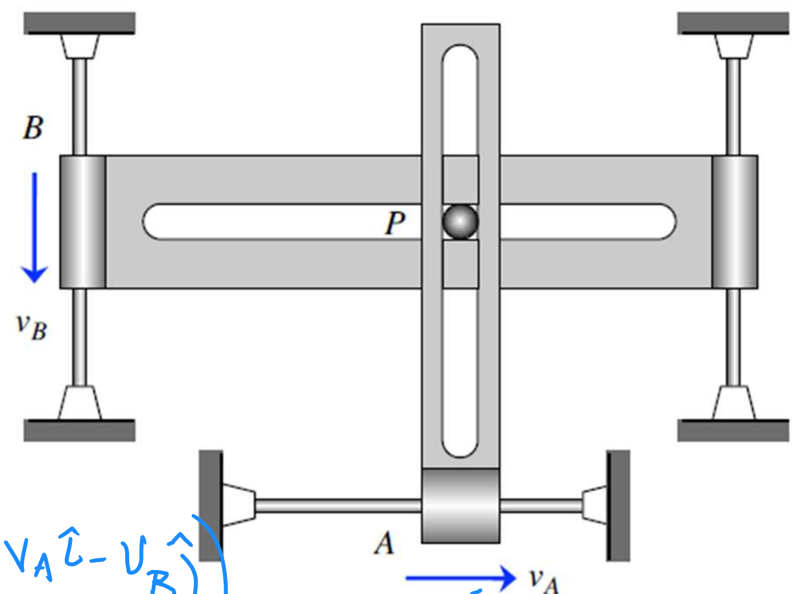
} path coord.

Use the following parameters: $v_A = 0.2$ m/s, $v_B = 0.15$ m/s, $\dot{v}_A = 0.75$ m/s^2 and $\dot{v}_B = 0$.

$$\vec{V} = \dot{x}\hat{i} + \dot{y}\hat{j} = V_A\hat{i} - V_B\hat{j}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = \dot{V}_A\hat{i} - \dot{V}_B\hat{j}$$

$$\vec{V} = V\hat{e}_t \rightarrow \hat{e}_t = \frac{\vec{V}}{V} = \frac{V_A\hat{i} - V_B\hat{j}}{\sqrt{V_A^2 + V_B^2}}$$



a) $$\vec{a} = \dot{V}\hat{e}_t + \frac{V^2}{\rho}\hat{e}_n$$

$$\dot{V} = \vec{a}\cdot\hat{e}_t = (\dot{V}_A\hat{i} - \dot{V}_B\hat{j})\cdot\left(\frac{V_A\hat{i} - V_B\hat{j}}{\sqrt{V_A^2 + V_B^2}}\right)$$

$$= \frac{1}{\sqrt{V_A^2 + V_B^2}}\left(\dot{V}_A V_A - \dot{V}_B V_B\right)$$

← sub & solve

Example 1.C.2.cont

b) $|a|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \rightarrow \rho = \frac{v^2}{\sqrt{|a|^2 - \dot{v}^2}}$

$|a|^2 = \dot{v}_A^2 + \dot{v}_B^2$

$\rightarrow \rho = \frac{v^2}{\sqrt{\dot{v}_A^2 + \dot{v}_B^2 - \dot{v}^2}}$ } from part A

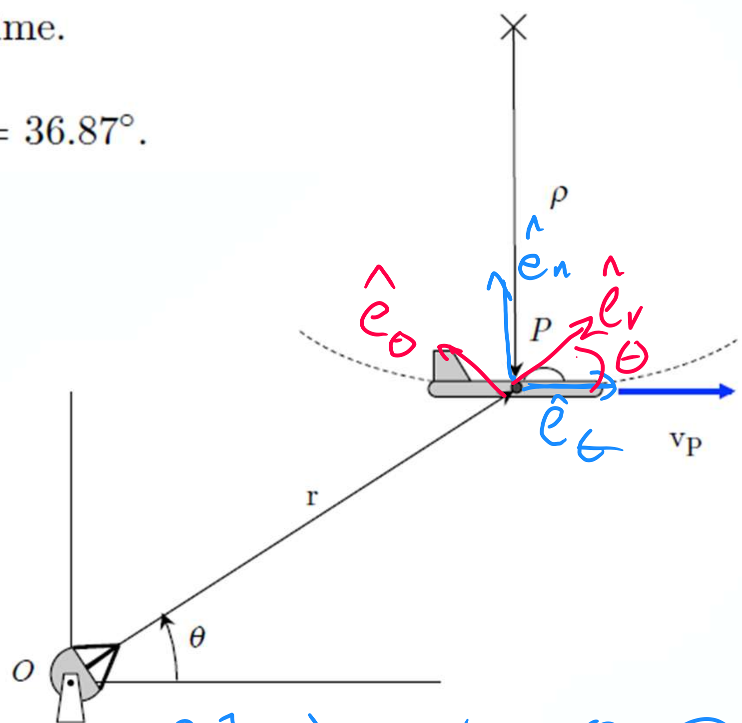
Example 1.C.4

$\dot{v}_p = 0$

Given: At the bottom of a loop, an airplane P has a constant speed of v_P with the radius of curvature for the aircraft being ρ. The airplane is at a radial distance of r and at an angle of θ from a radar tracking station at O.

Find: Determine numerical values for $\ddot{r}$ and $\ddot{\theta}$ at this instant in time.

Use the following: $v_P = 75$ m/s, $\rho = 3000$ m, $r = 1000$ m and $\theta = 36.87°$.



target: $\hat{e}_r, \hat{e}_\theta$ given: $\hat{e}_t, \hat{e}_n$

from geometry:

$$\hat{e}_r = \cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n$$

$$\hat{e}_\theta = -\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n$$

$$\vec{v} = v\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\dot{r} = \vec{v}\cdot\hat{e}_r = (v\hat{e}_t)\cdot(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n) = v\cos\theta$$

$$r\dot{\theta} = \vec{v}\cdot\hat{e}_\theta = (v\hat{e}_t)\cdot(-\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n) = -v\sin\theta$$

given θ, r sub & solve $\dot{r}, \dot{\theta}$

Example 1.C.4.cont

$$\vec{a} = \underbrace{\dot{v}}_{=0}\hat{e}_t + \underbrace{\frac{v^2}{\rho}}_{\text{given}}\hat{e}_n = \underline{(\ddot{r} - r\dot{\theta}^2)\hat{e}_r} + \underline{(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta}$$

$$\vec{a} = \frac{v^2}{\rho}\hat{e}_n$$

$$(\ddot{r} - r\dot{\theta}^2) = \vec{a}\cdot\hat{e}_r = \left(\frac{v^2}{\rho}\hat{e}_n\right)\cdot\left(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n\right)$$

$$(\underline{\ddot{r} - r}\dot{\theta}^2) = \frac{v^2}{\rho}\sin\theta$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \vec{a}\cdot\hat{e}_\theta = \left(\frac{v^2}{\rho}\hat{e}_n\right)\left(-\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n\right)$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{v^2}{\rho}\cos\theta$$

2 eqn, 2 unknowns → sub & solve for $\ddot{r}$, $\ddot{\theta}$