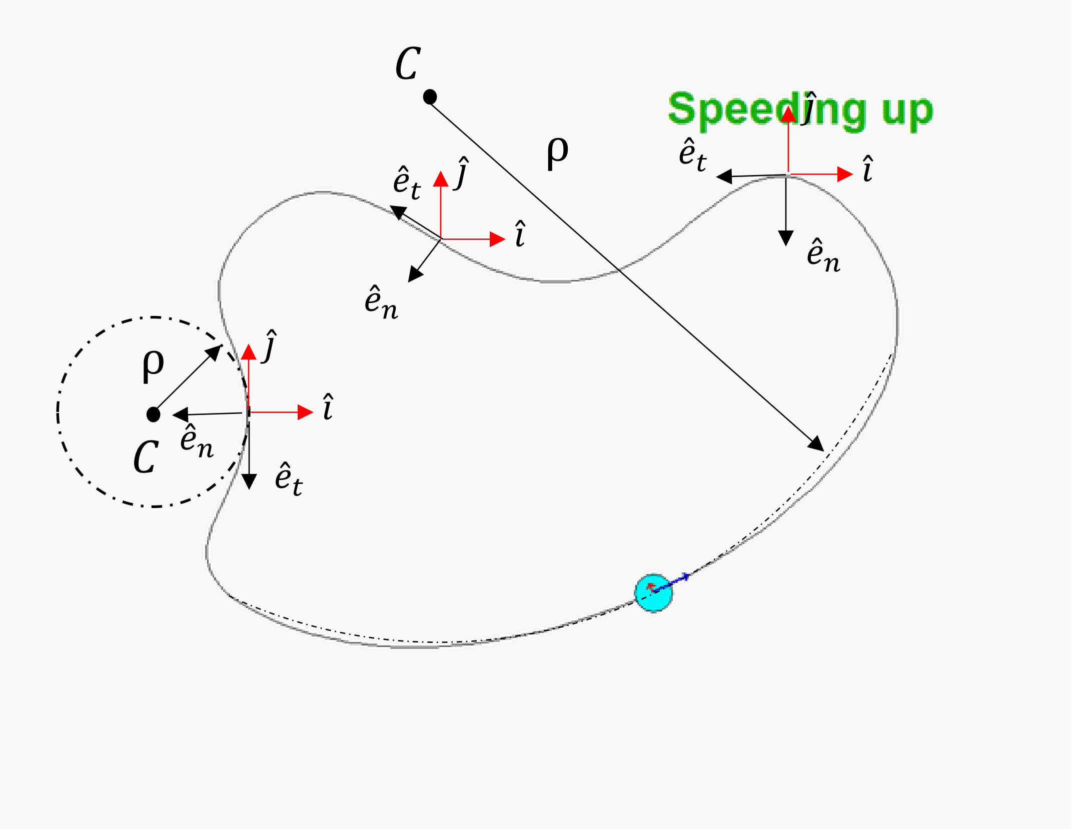
ME 274: Basic Mechanics II

Lecture 3: Point Kinematics, Polar Description

PURDUE UNIVERSITY | School of Mechanical Engineering

From Last Lecture...

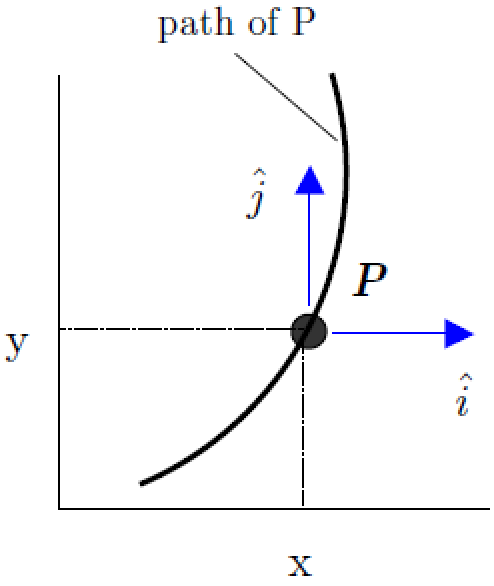
$$\vec{v} = v\hat{e}_t \qquad \vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$



Kinematic Descriptions

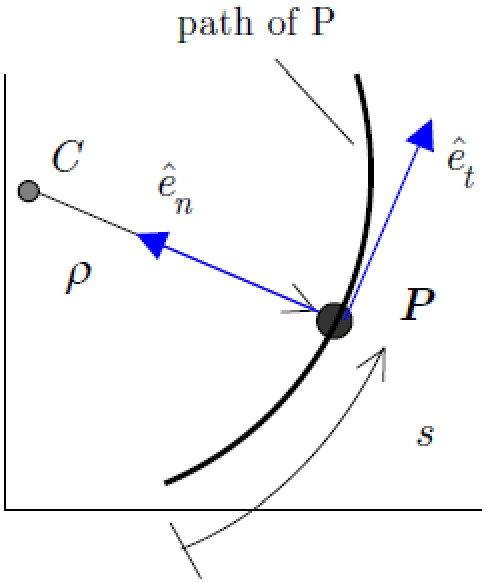
Cartesian description:

- Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath}$
- Fixed direction basis vectors $\hat{\imath}, \hat{\jmath}$
- $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$
- $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$



Path description:

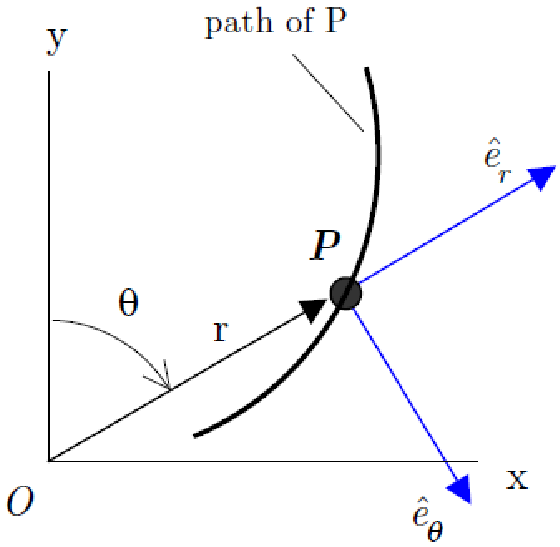
- Distance along path: $s(t)$
- $\hat{e}_t, \hat{e}_n$ depend on path geometry
- $\vec{v} = v\hat{e}_t$
- $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$



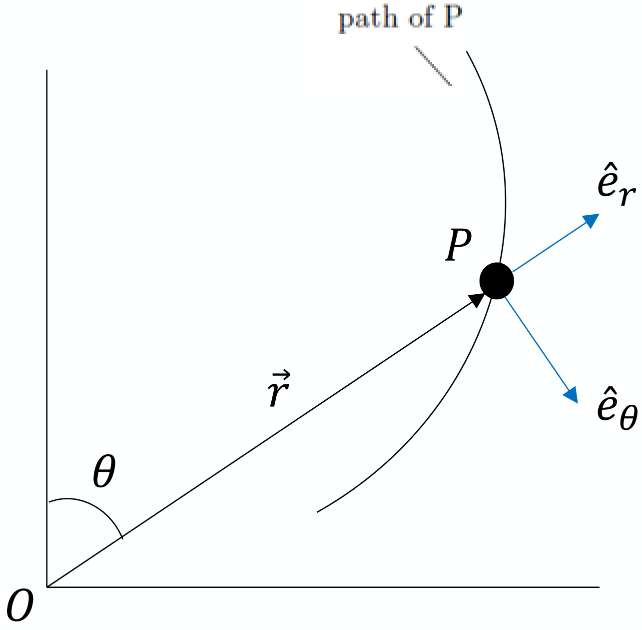
Today

Polar description:

- Position: $\vec{r} = r\hat{e}_r$
- $\hat{e}_r, \hat{e}_\theta$ change with particle motion
- $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
- $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r$ $+(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$



Polar Kinematics



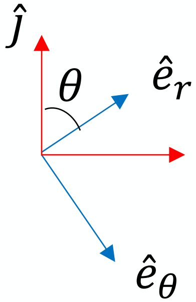
Position: $\vec{r} = r\hat{e}_r$

Velocity: $\vec{v} = \frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt}$$

Writing $\hat{e}_r$, $\hat{e}_\theta$ in terms of $\hat{\imath}, \widehat{\jmath}$:



$$\hat{e}_r = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_\theta = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\frac{d\hat{e}_r}{d\theta} = cos\theta\hat{\imath} - sin\theta\hat{\jmath} = \hat{e}_\theta$$

$$\boxed{\text{Velocity: } \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}$$

Polar Kinematics

Position: $\vec{r} = r\hat{e}_r$

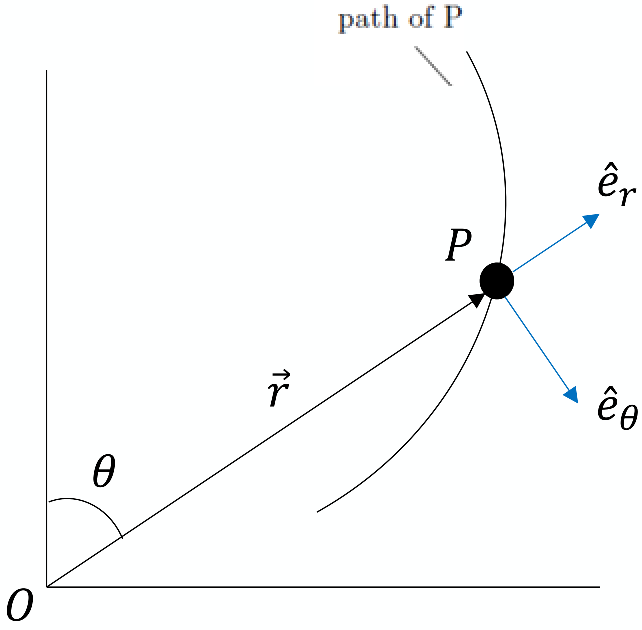
Velocity: $\vec{v} = \frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

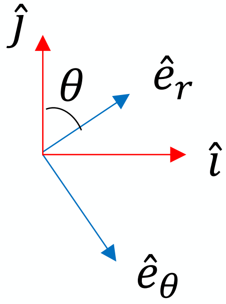
Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

$$= \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$$

$$\boxed{\dot{\theta}\hat{e}_\theta} \qquad \boxed{\frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt}}$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$





$$\hat{e}_r = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$\hat{e}_\theta = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

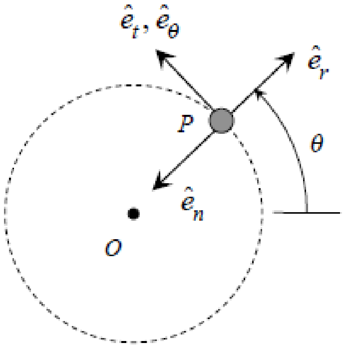
$$\Rightarrow$$

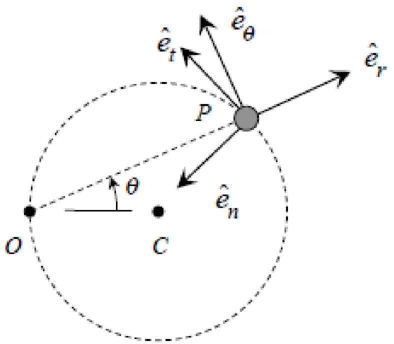
$$\frac{d\hat{e}_r}{d\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} = \hat{e}_\theta$$
$$\frac{d\hat{e}_\theta}{d\theta} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j} = -\hat{e}_r$$

$$\boxed{\begin{aligned}\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)\\ &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\end{aligned}}$$

Polar Kinematics – important considerations

- The values of the **vector components depend** on the location of your **reference point** and your **angle convention**

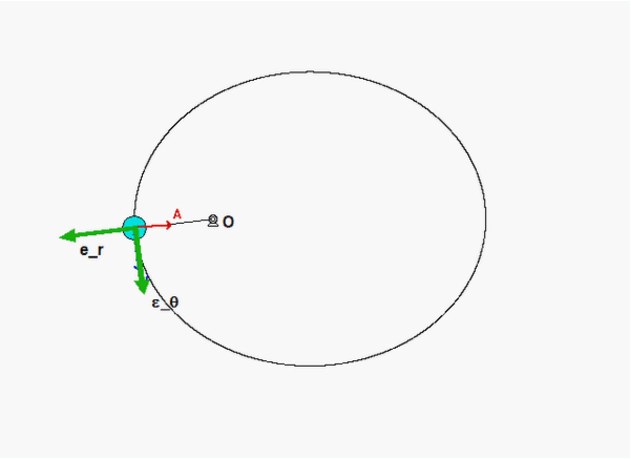




- When the path of P is given as $r = r(\theta)$, you will need the **chain rule** of differentiation to find time **derivatives in terms of $\dot{\theta}, \ddot{\theta}$**

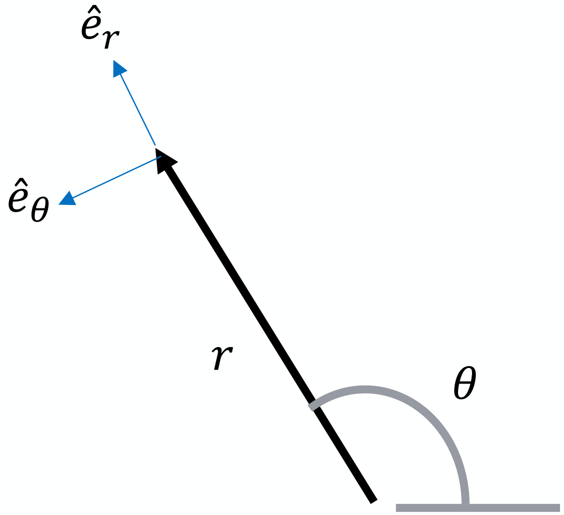
$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta}, \qquad \ddot{r} = \frac{d^2r}{d\theta^2}\dot{\theta} + \frac{dr}{d\theta}\ddot{\theta}$$

- Polar description of motion is useful in cases where there is an **observer of motion**



Conceptual question: C.6

Particle P travels on a path described by the polar coordinates $r = 2\cos\theta$, where r is given in feet and θ is given in radians. When $\theta = \pi/3$ radians, it is known that $\dot{\theta} = -3$ rad/s and $\ddot{\theta} = 0$.



At this instant

(a) $\dot{r} < 0$

(b) $\dot{r} = 0$

[✓ (c) $\dot{r} > 0$]

Solution:

$$\dot{r} = \frac{dr}{dt}$$

$$= \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(2cos\theta) = -2sin\theta$$

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\dot{r} = -2\dot{\theta}sin\theta = -2(-3)\sin\left(\frac{\pi}{3}\right) = 3\sqrt{3}\ ft/s$$

Conceptual question: C1.6

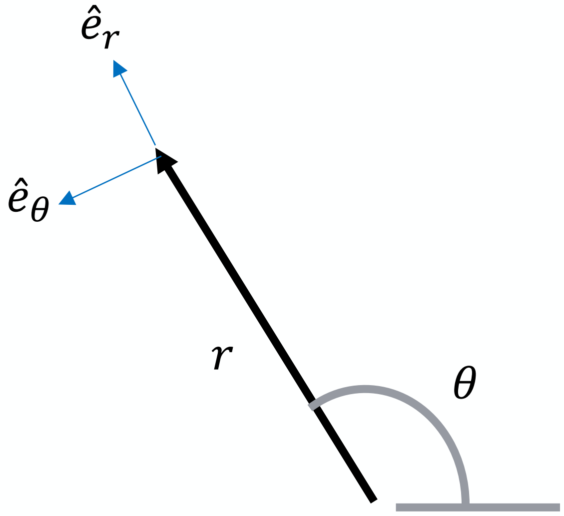
Particle P travels on a path described by the polar coordinates $r = 2\cos\theta$, where r is given in feet and θ is given in radians. When $\theta = \pi/3$ radians, it is known that $\dot{\theta} = -3$ rad/s and $\ddot{\theta} = 0$.

At this instant

[✓ (a) $\ddot{r} < 0$]

(b) $\ddot{r} = 0$

(c) $\ddot{r} > 0$



Solution:

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}(-2\dot{\theta}sin\theta)$$

$$= -2\left(\ddot{\theta}sin\theta + \dot{\theta}\frac{d}{dt}(sin\theta)\right)$$

$$\frac{d}{dt}(sin\theta) = cos\theta\dot{\theta}$$

$$\ddot{r} = -2\left(\ddot{\theta}sin\theta + \dot{\theta}^2cos\theta\right)$$

$$\Longrightarrow -2\left((0)\sin\left(\frac{\pi}{3}\right) + (-3)^2\cos\left(\frac{\pi}{3}\right)\right) = -9\ \text{ft/s}^2$$

Conceptual question: C1.8

Question C1.8

A particle P travels on a path given in terms of polar coordinates as: $r = \cos(3\theta)$, where r is given in feet, θ is given in radians and $\dot{\theta} = 2$ rad/s $=$ *constant*. Determine the magnitude of the acceleration vector of P when $\theta = \pi/3$.

Solution:

Remember $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\cancelto{0}{\ddot{\theta}} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$

Calculate derivatives of r:

$$r = \cos(3\theta) = \cos\left(\frac{3\pi}{3}\right) = -1\ ft$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}(-3sin3\theta) = 0\ ft/s$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = -3\left(\ddot{\theta}sin3\theta + 3\dot{\theta}^2cos3\theta\right) = -3\left((0)\sin(\pi) + 3(2)^2\cos(\pi)\right) = 36ft/s^2$$

Plug known values into expression for $\vec{a}$

$$\vec{a} = (36 - (-1)(2)^2)\hat{e}_r + \left((-1)(0) + 2(0)(2)\right)\hat{e}_\theta$$

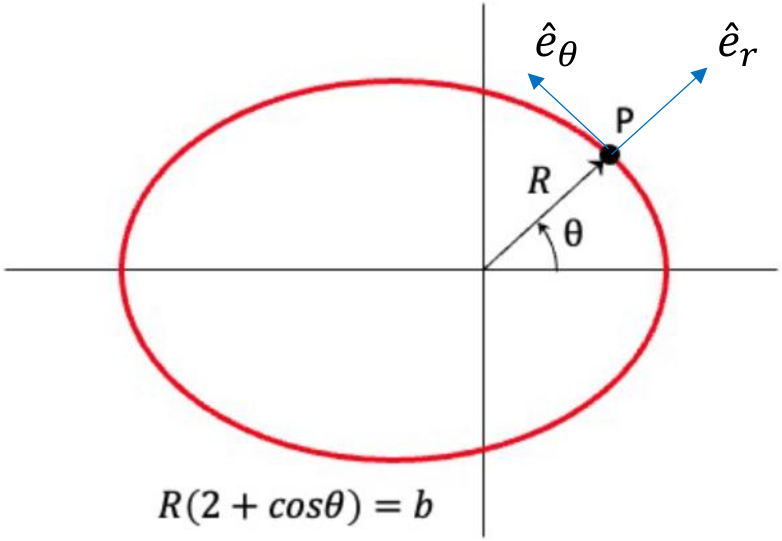
$$\vec{a} = 40\hat{e}_r\ ft/s^2$$

Additional lecture Example 1.3

Given: Particle P travels along an elliptical path shown with $\dot{\theta} = constant$

Find: for the position of P corresponding to $\theta = \frac{\pi}{2}$

a) Determine $\dot{R}$ and $\ddot{R}$. Use b = 2m, $\dot{\theta} = 3\frac{rad}{s}$



Solution:
Path defined by: $R(2 + cos\theta) = b$

At $\theta = \frac{\pi}{2}$,

$$R = \frac{b}{(2 + cos\theta)} = \frac{2}{2 + \cos\left(\frac{\pi}{2}\right)} = 1m$$

Solve for $\dot{R}$:

$$\frac{d}{dt}\text{:}\ \dot{R}(2 + cos\theta) + R\left(-\dot{\theta}sin\theta\right) = 0$$

Rearranging: $\dot{R} = \frac{\mathrm{R}\dot{\theta}sin\theta}{2+cos\theta}$

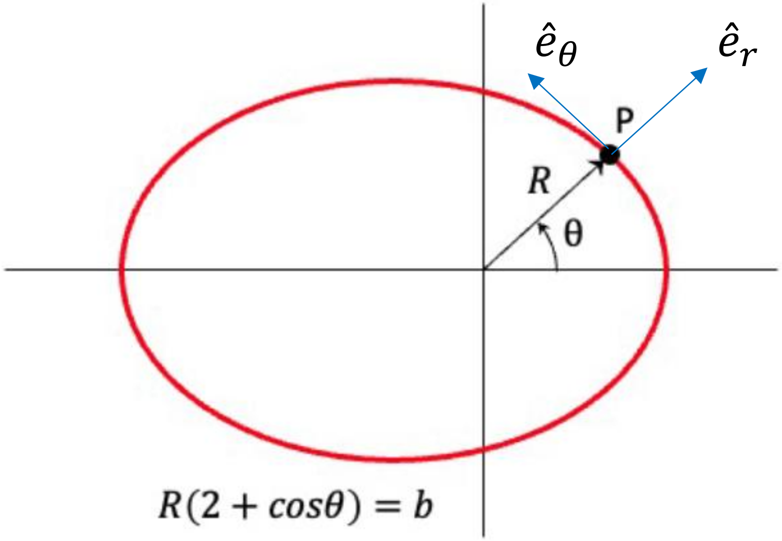
Plug in: $\dot{R} = \frac{(1)(3)\sin\left(\frac{\pi}{2}\right)}{2+\cos\left(\frac{\pi}{2}\right)} = \frac{3}{2}m/s$

Additional lecture Example 1.3

Given: Particle P travels along an elliptical path shown with $\dot{\theta} = constant$

Find: for the position of P corresponding to $\theta = \frac{\pi}{2}$

a) Determine $\dot{R}$ and $\ddot{R}$. Use b = 2m, $\dot{\theta} = 3\,\frac{rad}{s}$



Solution:

Path defined by: $R(2 + cos\theta) = b$

At $\theta = \frac{\pi}{2}$, $R = 1m, \dot{R} = \frac{3}{2}m/s$

Path eqn. derivatives:

$$\frac{d}{dt}: \dot{R}(2 + cos\theta) + R\left(-\dot{\theta}sin\theta\right) = 0$$

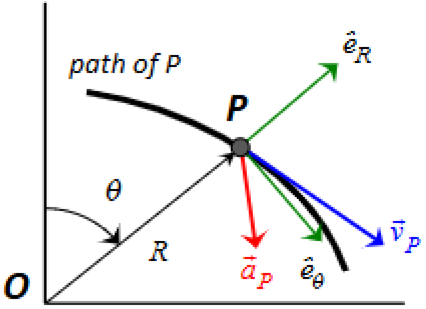
$$\frac{d^2}{dt^2}: \ddot{R}(2 + cos\theta) + \dot{R}\left(-\dot{\theta}sin\theta\right) + \dot{R}\left(-\dot{\theta}sin\theta\right)$$
$$+ R(-\ddot{\theta}sin\theta) + R(-\dot{\theta}^2cos\theta) = 0$$

Simplify: $2\ddot{R} - 2\dot{R}\dot{\theta} = 0$

Rearrange and solve: $\ddot{R} = \dot{R}\dot{\theta} = \frac{9}{2}m/s^2$

Summary: Particle Kinematics – Polar Description

1. *PROBLEM*: Motion of a point described in polar coordinates, *R* and θ.
2. *FUNDAMENTAL EQUATIONS*:



$$\vec{v}_P = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = velocity\ of\ P$$

$$\vec{a}_P = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta = acceleration\ of\ P$$

where $\hat{e}_R$ and $\hat{e}_\theta$ are the radial and transverse unit vectors.

3. *OBSERVATIONS: In regard to the polar description kinematics, we see*

- *You* are free to choose the observation point O.
- $\hat{e}_R$ *always* points OUTWARD from O to P. $\hat{e}_\theta$ is perpendicular to $\hat{e}_R$ and in direction of increasing θ.
- Polar description is useful for problems with observers or rotations about fixed axes.
- Do not confuse the unit radial vector $\hat{e}_R$ with the unit normal vector $\hat{e}_n$.