ME 274: Basic Mechanics II

Lecture 2: Point Kinematics, Path Description

PURDUE UNIVERSITY | School of Mechanical Engineering

Housekeeping

- Brightspace/Gradescope access?
- Lecture notes: https://www.purdue.edu/freeform/me274/quizzes-2/
- Hw 1 due **Tonight, 11:59 pm –**submit to Gradescope
- Hw 2 released today

From Last Lecture...

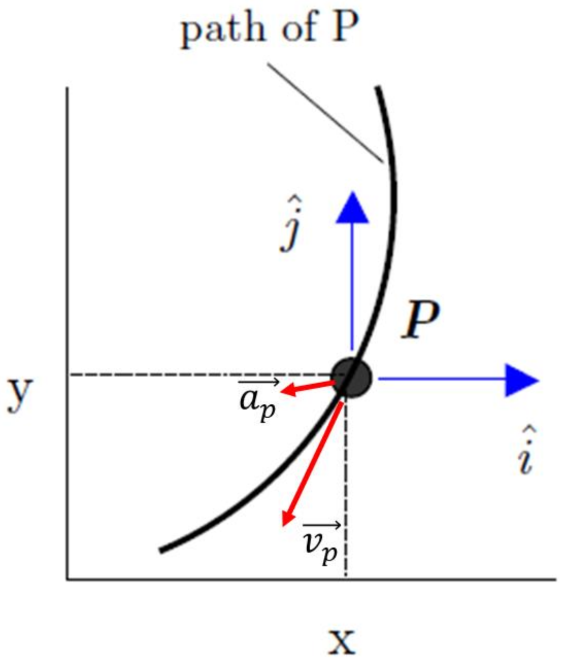
Describing motion: particle kinematics – for all coordinate systems:

- Position, $\vec{r}(t)$
- Velocity, $\vec{v}(t) \rightarrow \vec{v} = \frac{d\vec{r}}{dt}$
- Acceleration, $\vec{a}(t) \rightarrow \vec{a} = \frac{d^2\vec{r}}{dt^2}$

Cartesian description: The path of particle P is expressed in terms of ***x*** and ***y*** components

- $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$
- $\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$
- $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

Where $\hat{\imath}, \hat{\jmath}$ are **constant** basis vectors

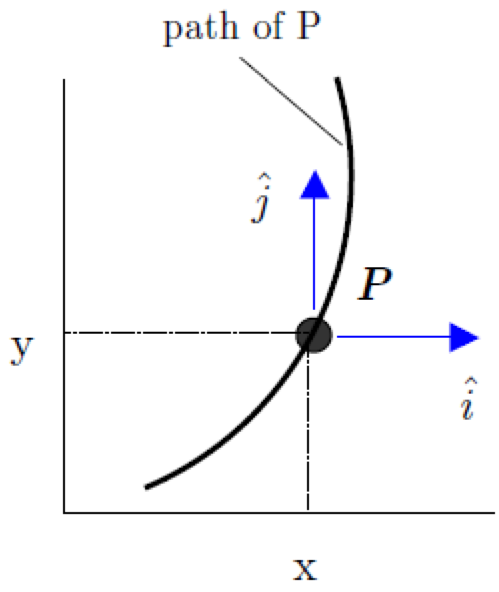


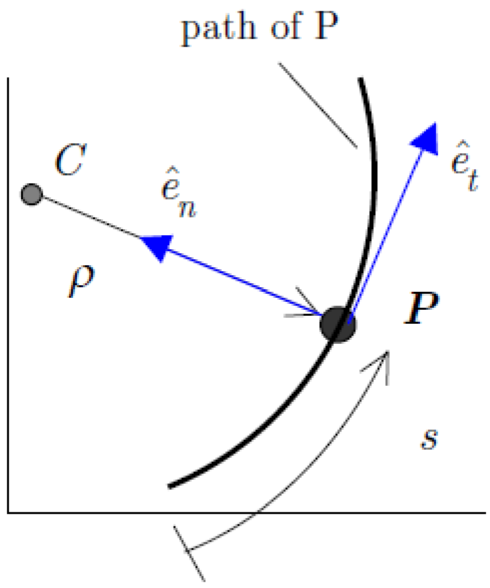
Differentiation requires the **chain rule:**

if $x = x(t)$ and $y = f(x)$:

- $\dot{x} = \frac{dx}{dt}$,
- $\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \dot{x}\frac{dy}{dx}$

Choosing a Kinematic Description





Cartesian description:

- Position: $x(t), y(t)$
- Fixed direction basis vectors $\hat{\imath}, \hat{\jmath}$

Works best when:

- Motion aligns with horizontal/vertical directions
- Geometry is naturally written in x and y

Path description:

- Position defined by s(t)
- Tangent and normal basis vectors $\hat{e}_t, \hat{e}_n$ depend on path geometry

Works best when:

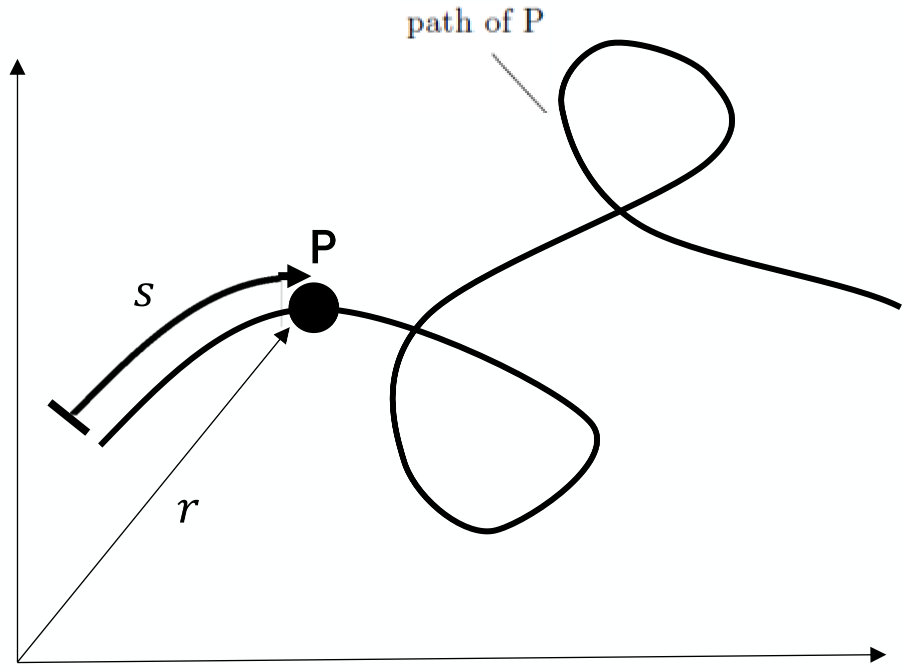
- Motion constrained to a known path
- We care about speed and curvature

Motivating Example: Roller Coasters

Path description: Position

If a particle, P, moves along a known path, the position, $\vec{r}$, is **known in terms of the distance**, s, the particle has moved **along the path.**

- $s = s(t) \leftarrow$ Scalar (arc length)
- $\vec{r} = \vec{r}(s) \leftarrow$ Vector



Remember for all descriptions

$$\vec{v} = \frac{d\vec{r}}{dt}$$

To solve for velocity in the path description, we use the **chain rule of differentiation:**

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

So what do these mean physically?

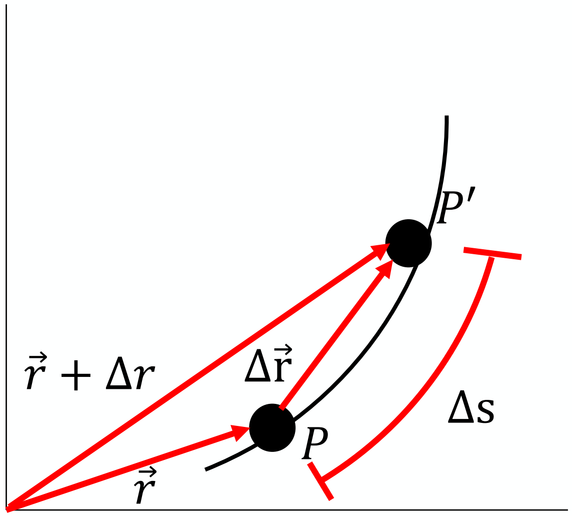
Velocity, Speed, and $\hat{e}_t$

$$\text{Velocity vector: } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

$\frac{ds}{dt}$ →the rate of change of distance with respect to time = **speed, $\boldsymbol{v}$**

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0}\left(\frac{\Delta\vec{r}}{\Delta s}\right)$$

- $\Delta\vec{r}$ ⟶ change in particle position
- $|\Delta\vec{r}|$⟶ chord length
- Δs ⟶ arc length



Magnitude of $\frac{d\vec{r}}{ds}$:

As $\Delta s \to 0$, $|\boldsymbol{\Delta\vec{r}}| \to \boldsymbol{\Delta s}$,

meaning $\frac{|\Delta\vec{r}|}{\Delta s} = 1$, making $\frac{d\vec{r}}{ds}$ a **unit vector**

Direction of $\frac{d\vec{r}}{ds}$:

As $\Delta s \to 0$, $\Delta\vec{r}$ becomes **tangent** to the path

Therefore: $\frac{d\vec{r}}{ds}$ **is a tangent unit vector, $\hat{e}_t$,** and $\vec{v} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \boldsymbol{v\hat{e}_t}$

Acceleration and curvature

Differentiate wrt time:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\ \hat{e}_t)$$

Use the product rule:

$$\vec{a} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{dt}$$

Use the chain rule:

$$\frac{d\hat{e}_t}{dt} = \frac{d\hat{e}_t}{ds}\frac{ds}{dt}$$ ←remember $\frac{ds}{dt} = v$

Using chain rule and path geometry:

$$\frac{d\hat{e}_t}{ds} = \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}$$

Therefore:

$$\vec{a} = \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}$$ ←we can simplify further

Acceleration and curvature

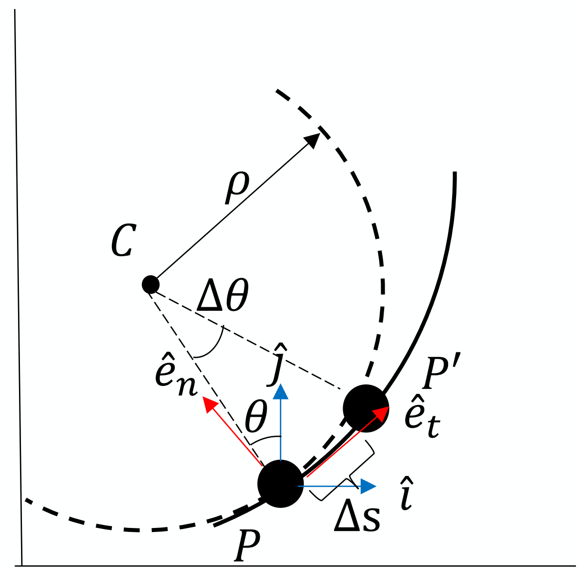
$$\vec{a} = \dot{v}\hat{e}_t + v^2 \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}$$

$\hat{e}_t$ and $\hat{e}_n$ can be related to $\hat{\imath}$ and $\hat{\jmath}$ as:

$$\hat{e}_n = -\sin\theta\,\hat{\imath} + \cos\theta\hat{\jmath}$$
$$\hat{e}_t = cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}$$

Therefore:

$$\frac{d\hat{e}_t}{d\theta} = \; -\sin\theta\,\hat{\imath} + \cos\theta\hat{\jmath} = \hat{e}_n$$



C = "center of curvature"

ρ = "radius of curvature"

From path geometry:

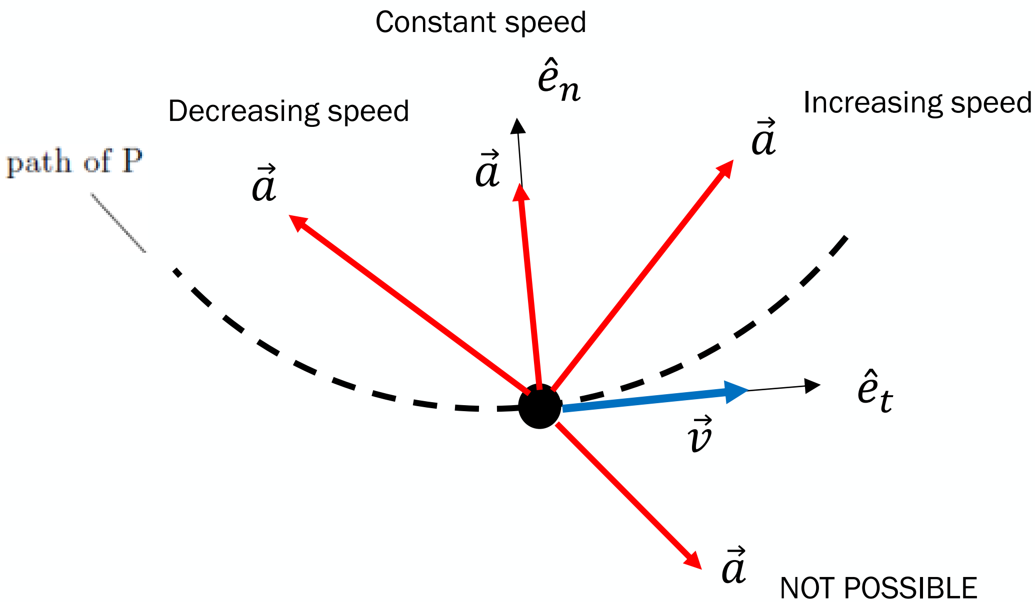
$$\mathrm{d}s = \rho\mathrm{d}\theta \;\; \Rightarrow \frac{ds}{d\theta} = \frac{1}{\rho}$$

Combining terms:

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Fundamentals of the Path Description

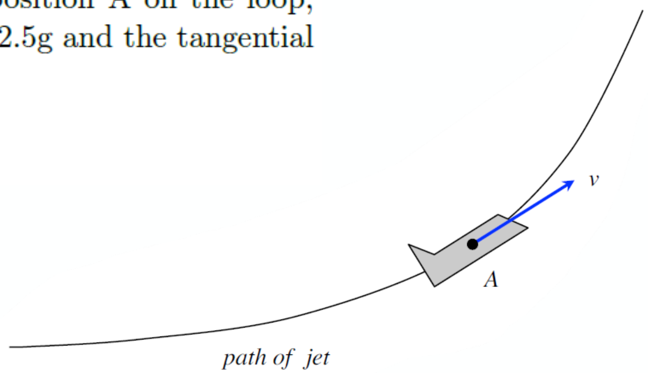
- Velocity: $\vec{v} = v\hat{e}_t$
- Acceleration: $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$
  - $\hat{e}_n$ is always pointed inward
  - $\dot{v} > 0 \longrightarrow$ particle is speeding up
  - $\dot{v} < 0 \longrightarrow$ particle is slowing down
  - $\dot{v} = 0 \longrightarrow$ particle speed is constant



Sample Problem: 1.A.3

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

Find: The radius of curvature of the path of the jet at A.



Solution:

Remember $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

We are given $|\vec{a}|$, $a_t = \dot{v}$, and v

Find an expression for $|\vec{a}|$:

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Rearrange and solve for ρ:

$$|\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2$$

$$\frac{v^2}{\rho} = \sqrt{|\vec{a}|^2 - \dot{v}^2}$$

Has to be positive

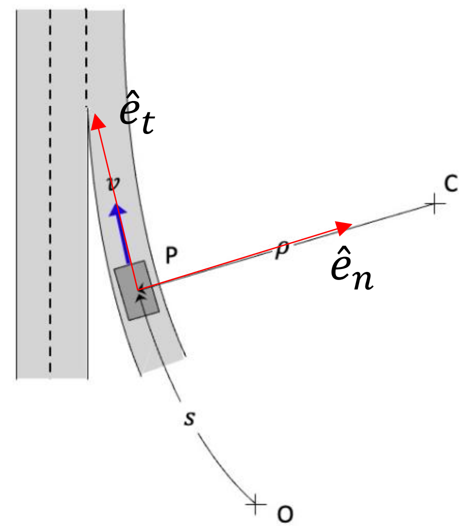
$$\rho = \frac{v^2}{\sqrt{|\vec{a}|^2 - \dot{v}^2}}$$

Sub in known values

Additional lecture Example 1.1

Given: An automobile P is entering a freeway along a "clothoid-shaped" entrance ramp whose radius of curvature ρ is given by $\rho = (a + bs)^{-1}$, where a and b are constants, and s is the distance traveled along the entrance ramp. The speed of P is known as a function of position s on the entrance ramp to be: $v(s) = c + ds$, where c and d are constants.

Use the following parameters in your work: $a = 0.005/\text{ft}$, $b = 1 \times 10^{-5}/\text{ft}^2$, $c = 25$ ft/s and $d = 0.25/\text{s}$.



Find:

(a) Determine the velocity and acceleration vectors for P. Express these vectors in terms of their path coordinates, and in terms of, at most: s, a, b, c and d.

Solution:

Velocity→ $\vec{v} = v\hat{e}_t = (c + ds)\hat{e}_t$

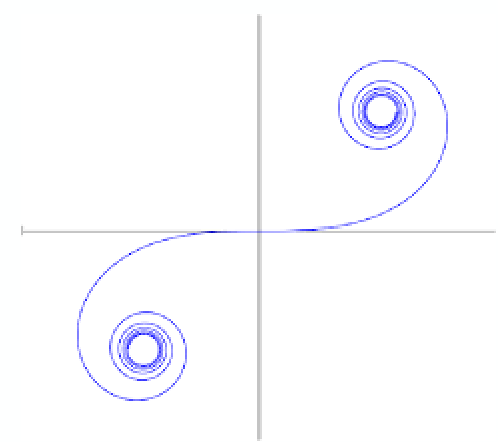
Acceleration → $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

v

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$$

$$\dot{v} = (c + ds)d$$

$$\frac{v^2}{\rho} = (c + ds)^2(a + bs)$$



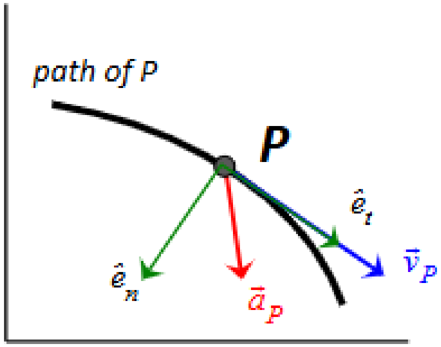
Clothoid spiral

$$\rightarrow \vec{a} = (c + ds)d\ \hat{e}_t + (c + ds)^2(a + bs)\ \hat{e}_n$$

Summary: Particle Kinematics – Path Description

1. *PROBLEM*: Motion of a point described in path variables.

2. *FUNDAMENTAL EQUATIONS*:



$$\vec{v}_P = v_P \hat{e}_t = velocity\ of\ P$$

$$\vec{a}_P = \dot{v}_P \hat{e}_t + \frac{v_P^2}{\rho} \hat{e}_n = acceleration\ of\ P$$

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors tangent and (inwardly) normal to the path.

3. *OBSERVATIONS: In regard to the path description kinematics, we see*

- Velocity is ALWAYS tangent to the path.
- Acceleration, in general, has BOTH normal and tangential components.
- Note that acceleration depends on three factors: speed v_P, rate of change of speed $\dot{v}_P$ and radius of curvature of the path ρ.
- Rate of change of speed is the projection of acceleration onto the unit tangent vector: $\dot{v}_P = \vec{a}_P \bullet \hat{e}_t$
- Rate of change of speed is NOT equal to the magnitude of acceleration:

$$|\vec{a}_P| = \sqrt{\dot{v}_P^2 + \left(v_P^2 / \rho\right)^2} \neq |\dot{v}_P|$$