ME 274 Lecture 2

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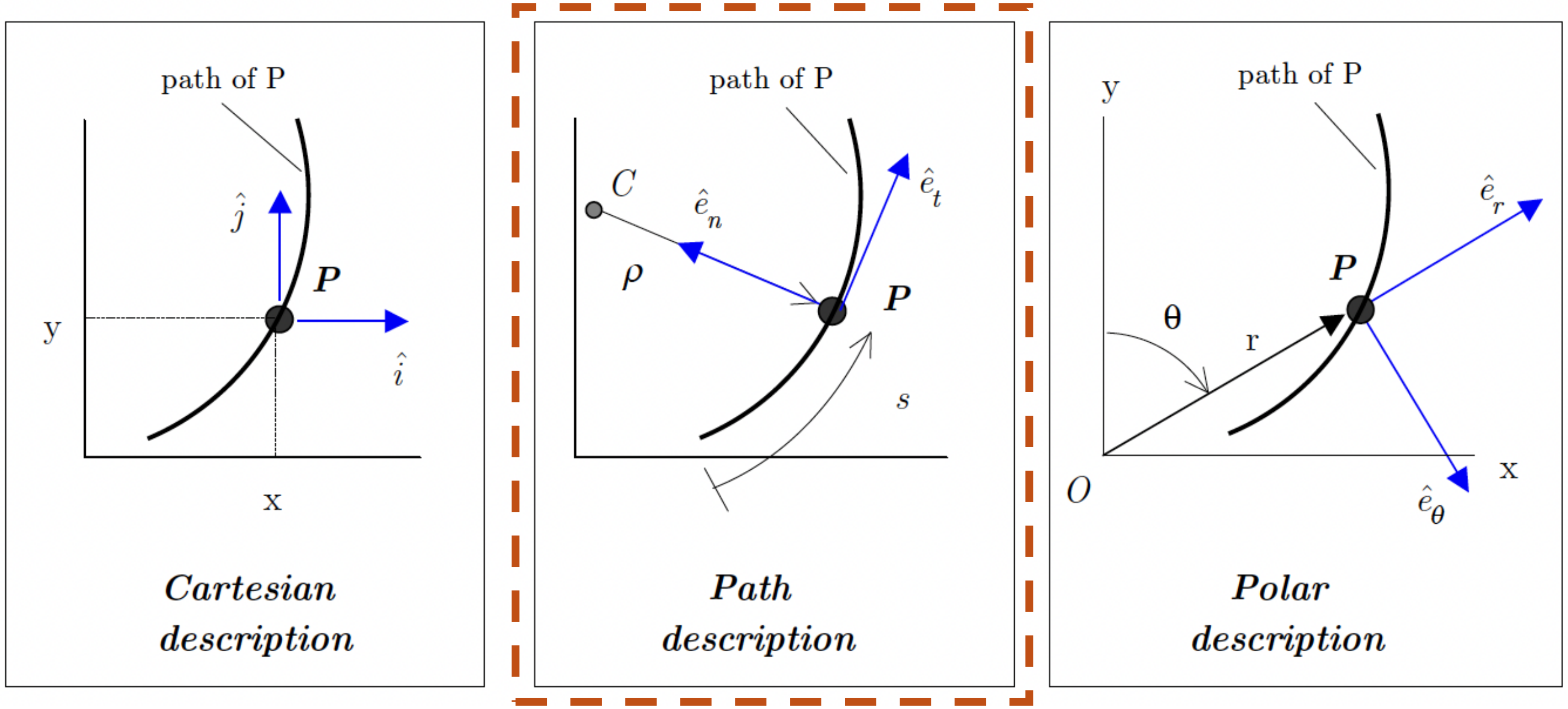
1/14/26

Point Kinematics – Overview

Today's Lecture:



[pg. 30]

The kinematics of velocity and acceleration may be described in Cartesian, path, or polar coordinates (figure above).

<u>The main difference between these is how we define our unit vectors</u>

Path Description's Unit Vectors

Tangential Unit Vector:

$$\hat{e}_t$$

- Unit Vector that is tangent to the path of P

Normal Unit Vector:

$$\hat{e}_n$$

- Points in the direction of the bend in the path of P (ie points to the center of the curve ~in ME274…)

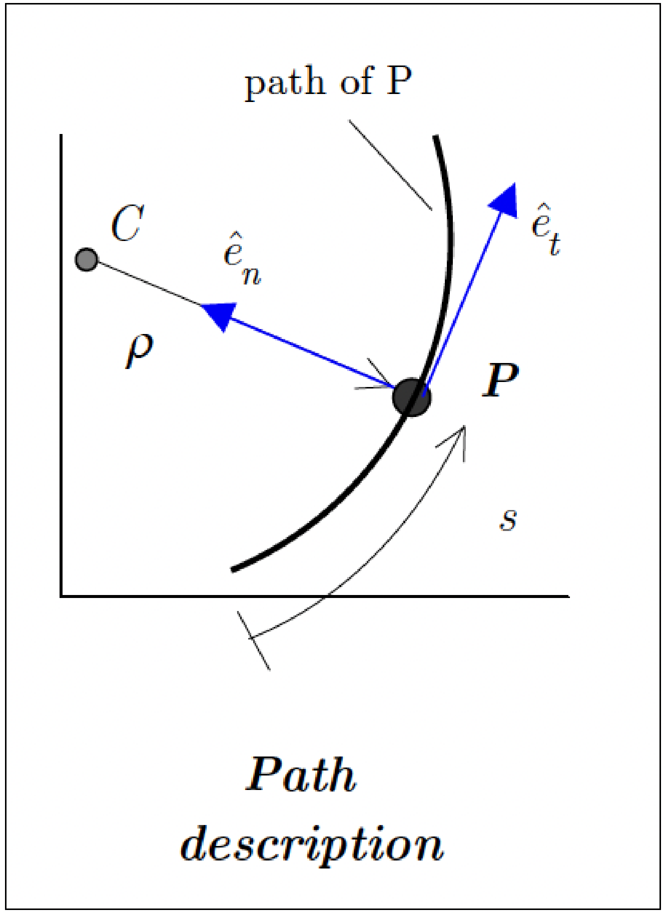
What we are building towards:

- Fundamental Equations of Today

$$\vec{v} = v\hat{e}_t$$

[pg. 34]

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$



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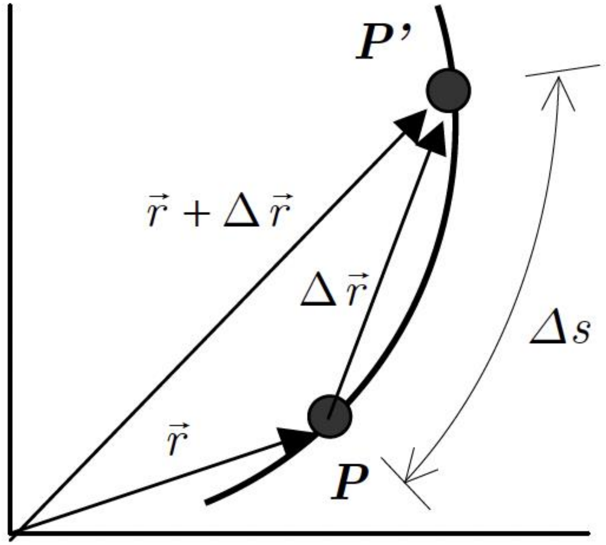
Where do we get path description from?

- For the path description, we start by looking at (1) r, the distance from the origin to P, and (2) s, the distance measured along the path of P.

- We take the derivative of r with respect to time (d/dt): $\vec{v} = \frac{d\vec{r}}{dt}$

- Using the chain rule, we end up with $\frac{d\vec{r}}{ds}\frac{ds}{dt}$ where $\frac{ds}{dt}$ is equal to v (speed of the particle)



- In summary: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\frac{d\vec{r}}{ds}$

[pg. 33 content]

Path description's *velocity vector* derivation

- Building on $v\frac{d\vec{r}}{ds}$; if we take the limit of $\frac{d\vec{r}}{ds}$ …

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\vec{r}}{\Delta s}$$

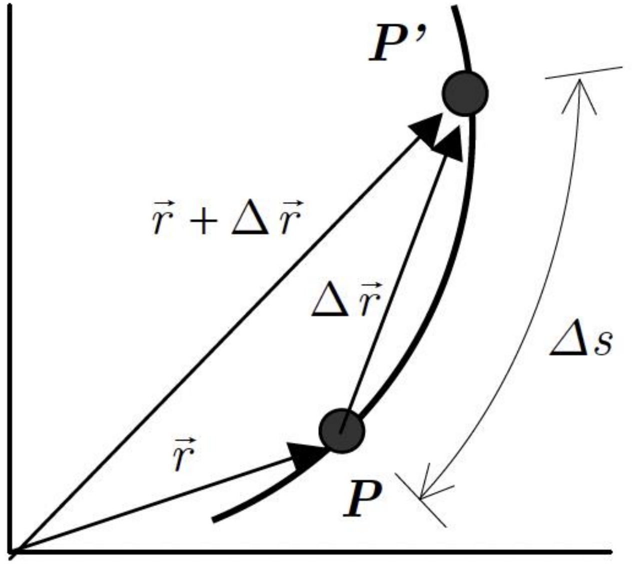
- Two observations can be made:

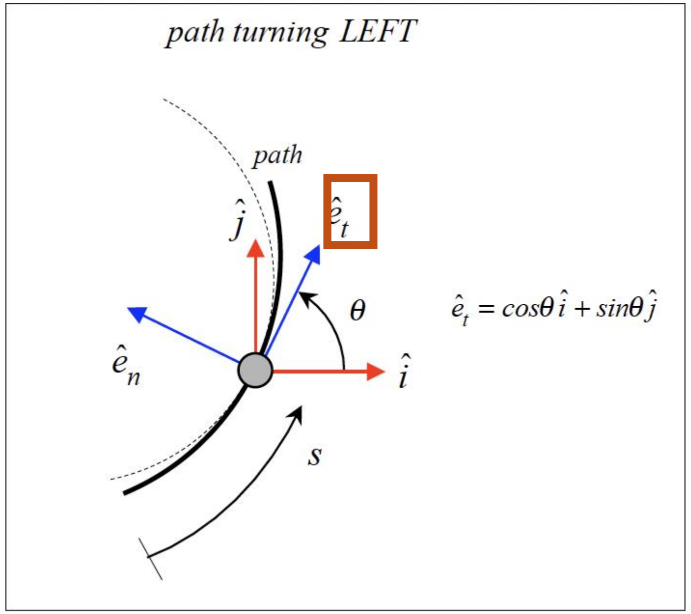
1. As delta(s) approaches zero, delta(r) tends to the length of the arc length delta(s).
Therefore, dr/ds is a ”unit vector”

2. As delta(s) approaches zero, the vector delta(r) becomes **tangent to the path of P**

- From this, we conclude that dr/ds = et = unit vector that is tangent to the path of P:

$$\vec{v} = v\hat{e}_t$$





[pg. 33 content]

Path description's *acceleration vector* derivation

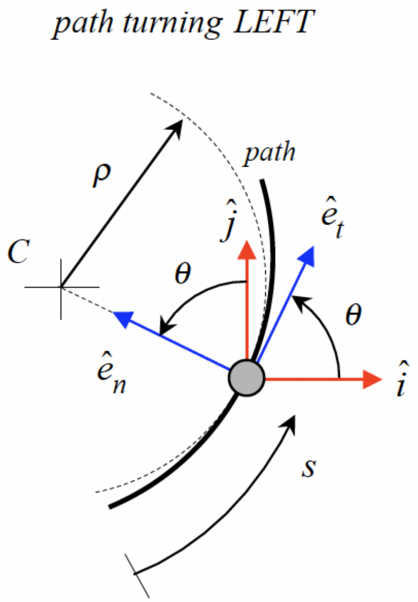
$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(v\hat{e}_t)$$

$$= \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \qquad \text{(product rule of differentiation)}$$

$$= \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{ds}\frac{ds}{dt} \qquad \text{(chain rule of differentiation)}$$

$$= \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} \qquad \left(\text{chain rule of differentiation and } v = \frac{ds}{dt}\right)$$



$$\hat{e}_n = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_t = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \Rightarrow$$

$$\frac{d\hat{e}_t}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_n$$

$$ds = +\rho\, d\theta \quad \Rightarrow \quad \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\therefore \quad \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

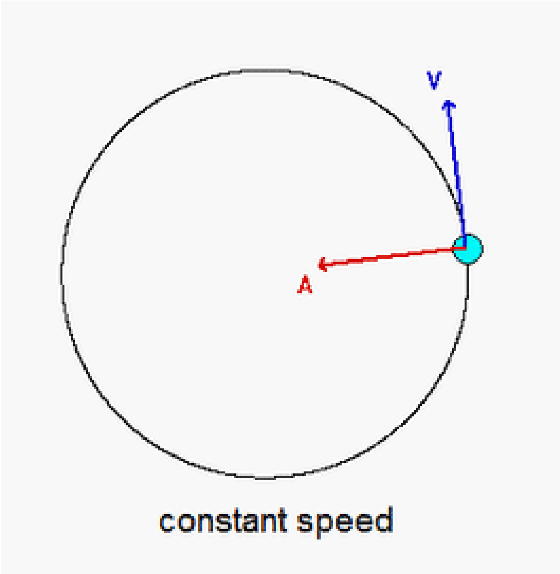
- We take the derivative of the velocity vector (in path description) and get…

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

[pg. 34-35 content]

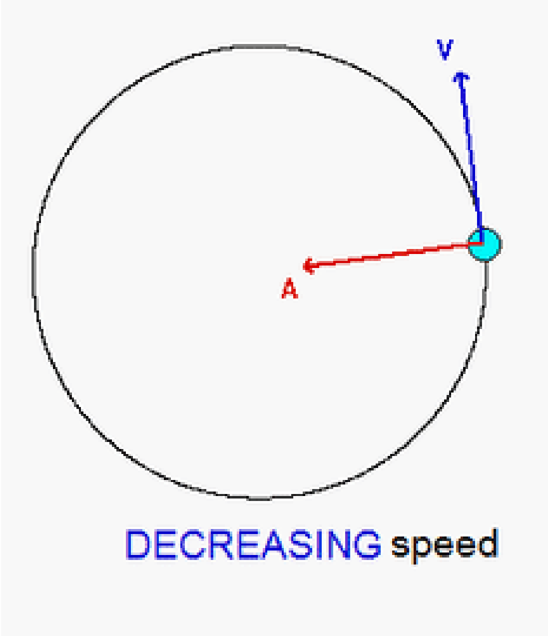
Path description – things to note

The velocity of a point is ALWAYS tangent to the path of the point. The magnitude of the velocity vector is the known as the scalar "speed" v of the point.



The acceleration of the point has two components:

- The component $(v^2/\rho)\,\hat{e}_n$ is *normal* to the path. This is commonly referred to as the "centripetal" component of acceleration. This component is ALWAYS directed inward to the path (*positive n-component*) since $v^2/\rho > 0$.
- The component $\dot{v}\hat{e}_t$ is tangent to the path. The magnitude of this component is the "rate of change of speed" $\dot{v}$ for the point.
  - When $\dot{v} > 0$ (increasing speed), the acceleration vector has a *positive t-component* (i.e., forward of $\hat{e}_n$).
  - When $\dot{v} = 0$ (constant speed), the acceleration vector has a *zero t-component* (i.e., $\vec{a}$ is aligned with $\hat{e}_n$). Note that constant speed does NOT imply zero acceleration!
  - When $\dot{v} < 0$ (decreasing speed), the acceleration vector has a *negative t-component* (i.e., backward of $\hat{e}_n$). See figure below.



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Path description – things to note (cont.)

- The magnitude of the acceleration is given by the square root of the sum of the squares of its path components:

$$|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$$

  – The magnitude of acceleration $|\vec{a}|$ accounts for both the tangential and normal components of acceleration, as shown in the above equation.

Do not confuse the terms "rate of change of speed" and "magnitude of acceleration":

– Rate of change of speed $\dot{v}$ (as the name indicates) is the rate at which the speed changes in time; it is simply the tangential component of acceleration.

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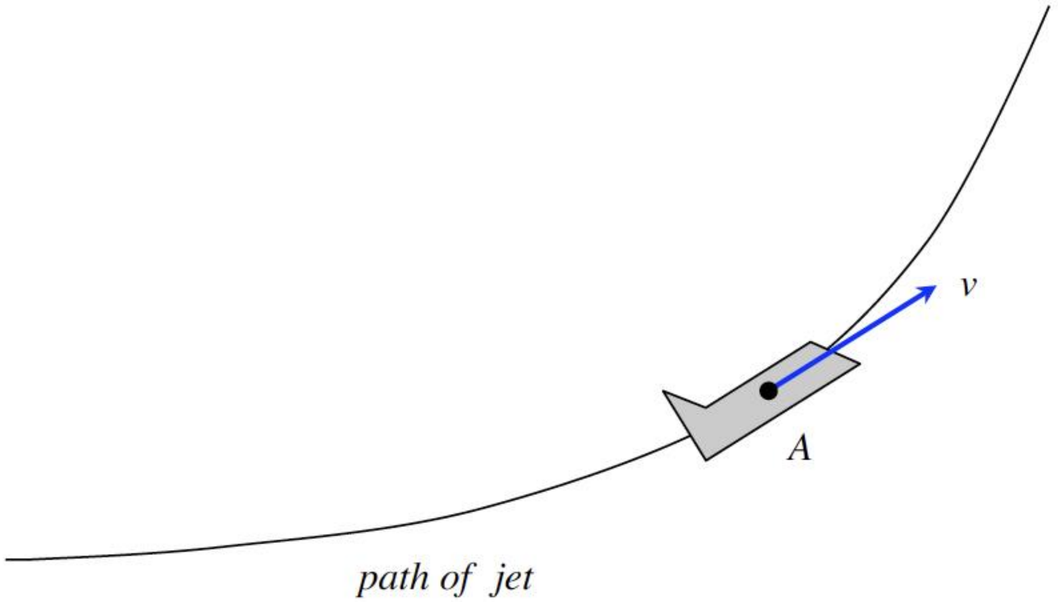
Example 1.A.3

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

Find: The radius of curvature of the path of the jet at A.



[pg. 43]

Additional lecture Example 1.1

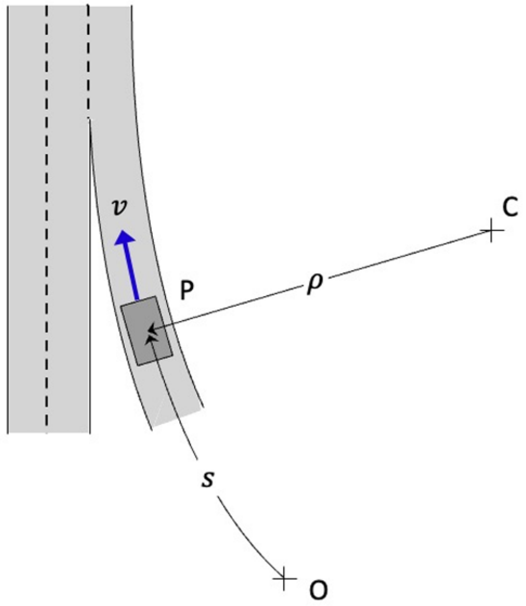
Given: An automobile P is entering a freeway along a "clothoid-shaped" entrance ramp whose radius of curvature ρ is given by $\rho = (a + bs)^{-1}$, where a and b are constants, and s is the distance traveled along the entrance ramp. The speed of P is known as a function of position s on the entrance ramp to be: $v(s) = c + ds$, where c and d are constants.

Find:

(a) Determine the velocity and acceleration vectors for P. Express these vectors in terms of their path coordinates, and in terms of, at most: s, a, b, c and d.

(b) Determine the numerical values of the velocity and acceleration vectors for P at the position at $s = 200$ ft.

(c) Make a sketch of these velocity and acceleration vectors, including the path unit vectors $\hat{e}_t$ and $\hat{e}_n$.



Use the following parameters in your work: $a = 0.005/\text{ft}$, $b = 1 \times 10^{-5}/\text{ft}^2$, $c = 25$ ft/s and $d = 0.25/\text{s}$.

[Additional Book Example for today]

Additional lecture Example 1.2

Given: Particle P moves along a path with its position on the path given by the arc length of s. The speed of P is given as a function of s as: $v_P = bs^2$, where s is given in meters and v_P in terms of meters/second. The radius of curvature of the path is given by ρ and the path tangent is at an angle of θ with respect to the direction of the x-axis.

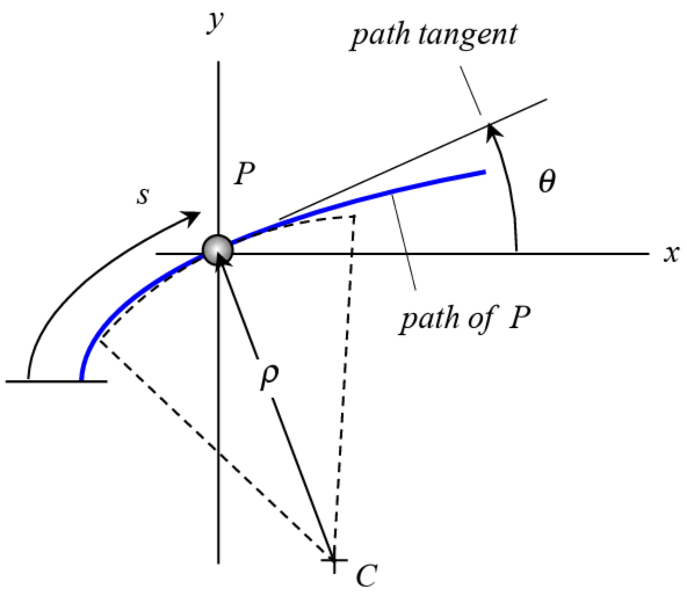
Find: At the position of P where $s = 3$ m:

(a) Make a sketch of the path unit vectors $\hat{e}_t$ and $\hat{e}_n$.

(b) Determine the velocity and acceleration of P in terms of path unit vectors $\hat{e}_t$ and $\hat{e}_n$.

(c) Determine the velocity and acceleration of P in terms of Cartesian unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

(d) Determine the xy-components of location of the center of curvature, C, for the path.



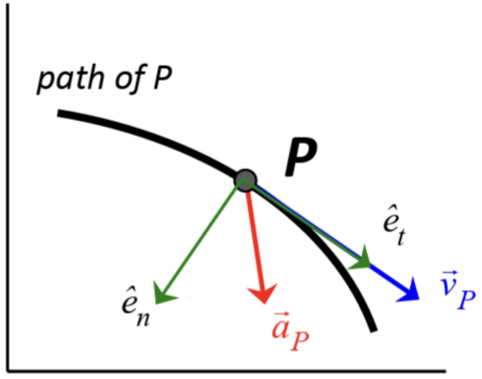
Use the following parameters in your work: $b = 0.5$/m-s, $\rho = 5$ m and $\theta = 30°$.

[Additional Book Example for today]

Summary: Particle Kinematics – Path Description

1. *PROBLEM*: Motion of a point described in path variables.

2. *FUNDAMENTAL EQUATIONS*:



$$\vec{v}_P = v_P\hat{e}_t = velocity\ of\ P$$ [pg. 33]

$$\vec{a}_P = \dot{v}_P\hat{e}_t + \frac{v_P^2}{\rho}\hat{e}_n = acceleration\ of\ P$$ [pg. 35]

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors tangent and (inwardly) normal to the path.

3. *OBSERVATIONS: In regard to the path description kinematics, we see*

- Velocity is ALWAYS tangent to the path. [pg. 36]
- Acceleration, in general, has BOTH normal and tangential components. [pg. 36 & challenge q on 37]
- Note that acceleration depends on three factors: speed v_P, rate of change of speed $\dot{v}_P$ and radius of curvature of the path ρ. [pg. 37]
- Rate of change of speed is the projection of acceleration onto the unit tangent vector: $\dot{v}_P = \vec{a}_P \bullet \hat{e}_t$
- Rate of change of speed is NOT equal to the magnitude of acceleration:

$$|\vec{a}_P| = \sqrt{\dot{v}_P^2 + \left(v_P^2/\rho\right)^2} \neq |\dot{v}_P|$$ [pg. 37]

Lec 2 Short Feedback Form:

Example 1.A.3 pg. 43

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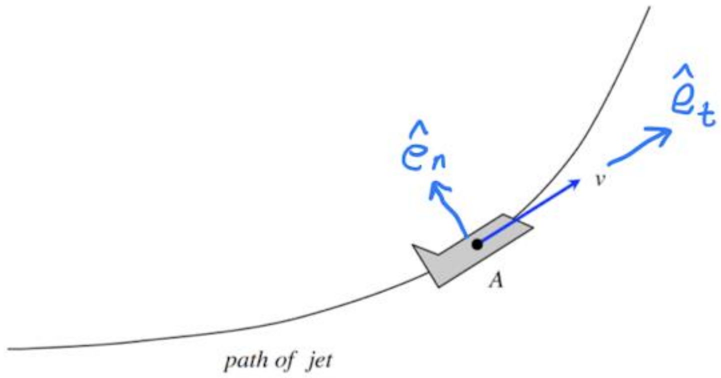
Find: The radius of curvature of the path of the jet at A.

$\hat{e}_{tangent} = \hat{e}_t$

$\hat{e}_{normal} = \hat{e}_n$

$\wedge$ = unit vector

$\rightharpoonup$ = vector



$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$= a_t\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$|\vec{a}|^2 = a_t^2 + \frac{v^4}{\rho^2}$$

$$\rho = \sqrt{\frac{v^4}{|\vec{a}|^2 - a_t^2}}$$

Additional lecture Example 1.1   In course website

Given: An automobile P is entering a freeway along a "clothoid-shaped" entrance ramp whose radius of curvature ρ is given by $\rho = (a + bs)^{-1}$, where a and b are constants, and s is the distance traveled along the entrance ramp. The speed of P is known as a function of position s on the entrance ramp to be: $v(s) = c + ds$, where c and d are constants.

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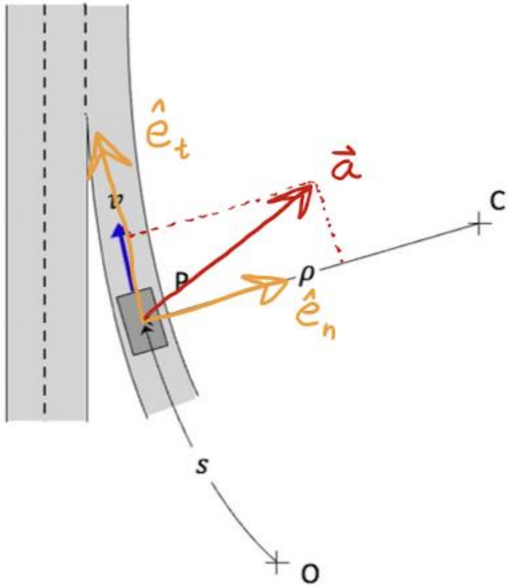

$$\vec{v} = v\,\hat{e}_t = (c + ds)\,\hat{e}_t$$

$$\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

w/ $$\dot{v} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = (c + ds)d$$

($\frac{ds}{dt} = v$; chain rule)

$$\vec{a} = \underbrace{(c+ds)d}_{>0}\,\hat{e}_t + \underbrace{(c+ds)^2(a+bs)}_{>0}\,\hat{e}_n$$

Additional lecture Example 1.2 In course website

Given: Particle P moves along a path with its position on the path given by the arc length of s. The speed of P is given as a function of s as: $v_P = bs^2$, where s is given in meters and v_P in terms of meters/second. The radius of curvature of the path is given by ρ and the path tangent is at an angle of θ with respect to the direction of the x-axis.

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(c) Determine the velocity and acceleration of P in terms of Cartesian unit vectors $\hat{i}$ and $\hat{j}$.

(d) Determine the xy-components of location of the center of curvature, C, for the path.

Use the following parameters in your work: $b = 0.5$/m-s, $\rho = 5$ m and $\theta = 30°$.

(b)

✓ $\vec{v}_p = v_p \hat{e}_t = (bs^2)\hat{e}_t$

✓ $\vec{a}_p = \dot{v}_p \hat{e}_t + \frac{v_p^2}{\rho}\hat{e}_n$

$= \dot{v}_p \hat{e}_t + \frac{(bs^2)^2}{\rho}\hat{e}_n$

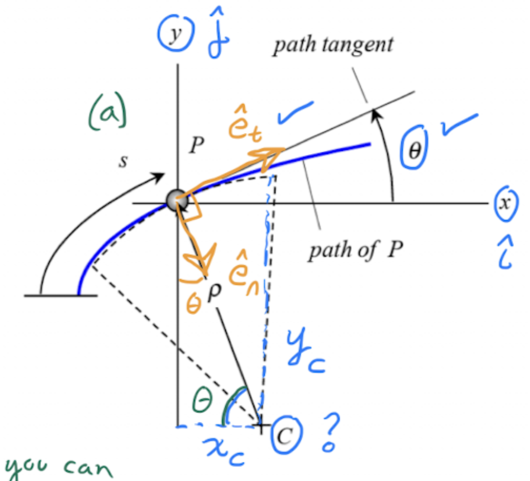
w/ $\dot{v}_p = \frac{dv_p}{dt} = \frac{dv_p}{ds}\frac{ds}{dt} = v_p \frac{dv_p}{ds}$; • chain rule ($\frac{ds}{dt} = v_p$)

$= (bs^2)(2bs)$

$= 2b^2s^3$

$=$ ____

w/ this you can plugin & solve for (b)



(c)

From figure: $\hat{e}_t = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_n = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

∴

$\vec{v}_p = bs^2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$ ⟵ (c)'s $\vec{v}_p$

$\vec{a}_p = 2b^2s^3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + \frac{(bs^2)^2}{\rho}(\sin\theta\,\hat{i} - \cos\theta\,\hat{j})$

$= \left[2b^2s^3\cos\theta + \frac{(bs^2)^2}{\rho}\sin\theta\right]\hat{i} + \left[2b^2s^3\sin\theta - \frac{(bs^2)^2}{\rho}\cos\theta\right]\hat{j}$ ⟵ (c)'s $\vec{a}_p$

(d) $x_c = \rho\sin\theta$, $y_c = -\rho\cos\theta$