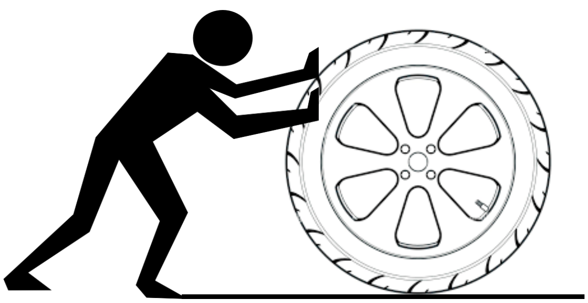
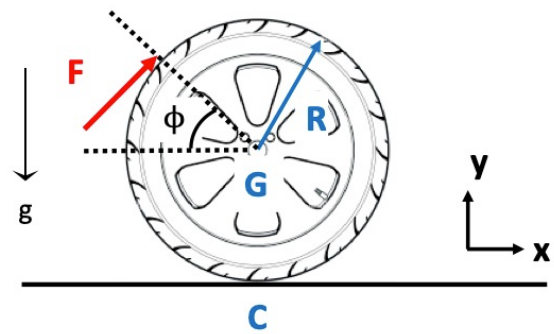
ME 274: Basic Mechanics II – Spring 2024 **Name**______________________________
Final Exam – April 29

Problem 1 *(20 points)*
A person is rolling a wheel with an outer radius of *R* on a flat ground. A constant force *F* is applied to the wheel, with its direction **tangent to** the wheel, and with the angle between the horizontal and the radial line at the force location is ϕ. The direction and acting position of the force *F* is not changing with time (i.e., $\boldsymbol{\phi}$ **is constant during the process**). The wheel is rolling without slipping at the contact point *C*. The mass of the wheel is *m* and the mass moment of inertia about its center of mass *G* is I_G.

Follow the solution steps below to find the acceleration of G and the angular velocity of the wheel at a prescribed instant in time.

Please use the following values in the calculation for the questions below: $F = 160$ N, $\phi = 30°$, $R = 0.5$ m, $I_G = 4\ \mathrm{kg \cdot m^2}$, $m = 32$ kg

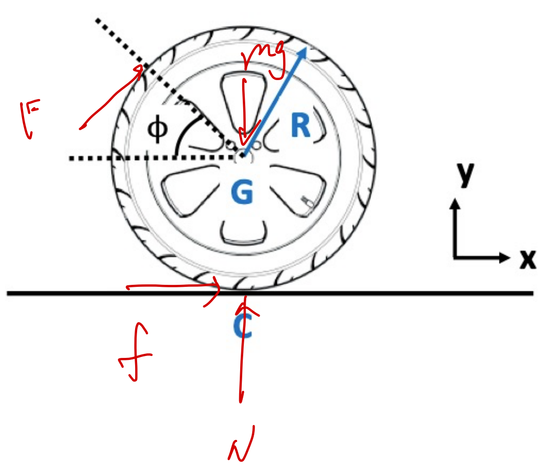




- **F is tangential to the wheel**
- **Rolling without slipping at C**

SOLUTION

(a) Draw free body diagram on the figure below



Problem 1 *(continued)*

(b) Find acceleration of the wheel center G. Express the answer in vector form using the coordinates given in the figure in Part (a).

Newton : $\sum F_x = m a_G$: $F\sin\phi + f = m a_G$

Euler : method ① : choose G :

$$\sum \vec{M}_G = I_G \vec{\alpha} \; : \; -FR + fR = I_G \alpha$$

method ② : choose C :

$$\sum \vec{M}_C = I_C \vec{\alpha} \; : \; -F\sin\phi\,(R + R\sin\phi) - F\cos\phi\, R\cos\phi = I_C \alpha$$

$$I_C = I_G + mR^2$$

Kinematics : $\vec{a}_G = \vec{a}_C + \vec{\alpha} \times \vec{r}_{G/C} - \omega^2 \vec{r}_{G/C}$

$\hat{i}$: $a_G = -R\alpha$

Solve : $\alpha = \dfrac{-FR(1+\sin\phi)}{I_G + mR^2} = -10 \text{ rad/s}^2$,

$\boxed{a_G = -R\alpha = 5 \text{ m/s}^2}$ $\qquad f = m a_G - F\sin\phi = 80 \text{ N}$

(c) At the initial time of $t = 0$, the wheel is at rest. Find the wheel's angular speed ω at $t = 5$ s.

method ① : $\boxed{\omega_2 = 0 + \int_0^5 \alpha\, dt = -50 \text{ rad/s}}$

method ② : $v_2 = 0 + \int_0^5 a_G\, dt = 25 \text{ m/s}$

Kinematics : $v = -R \cdot \omega \Rightarrow \boxed{\omega_2 = -\dfrac{v_2}{R} = -50 \text{ rad/s}}$

method ③ : Angular momentum, choose C :

$H_2 = 0 + \int_0^5 M_C\, dt = \int_0^5 -FR(1+\sin\phi)\, dt = -120 \times 5 = -600 \text{ kg}\cdot\text{m}^2/\text{s}$

$\boxed{\omega_2 = \dfrac{H_2}{I_C} = -50 \text{ rad/s}}$

method ④ : Angular momentum : choose G :

$H_2 = 0 + \int_0^5 M_G\, dt = \int_0^5 (-FR + fR)\, dt = -40 \times 5 = -200 \text{ kg m}^2/\text{s}$

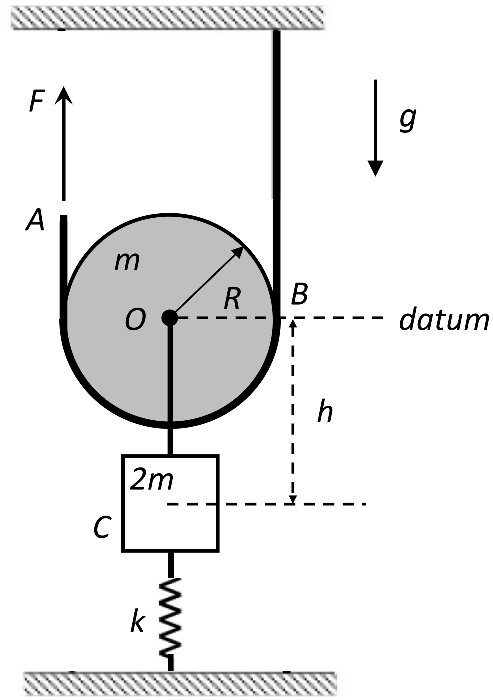
$\boxed{\omega_2 = \dfrac{H_2}{I_G} = -50 \text{ rad/s}}$

method ⑤ : linear momentum : $m v_2 = 0 + \int_0^5 (F\sin\phi + f)\, dt = 160 \times 5 = 800 \text{ kg m/s}$

$\Rightarrow v_2 = 25 \text{ m/s} \Rightarrow \boxed{\omega_2 = -\dfrac{v_2}{R} = -50 \text{ rad/s}}$

Problem 2 *(20 points)*

Given: A homogeneous pulley (having a mass m and outer radius R) is supported by a cable. A block C with mass $2m$ is connected to the pulley with a rope at point O. The height difference between the pulley and block is h. The block is connected to the spring with a stiffness of k and negligible mass. The system is released from rest with the spring initially unstretched (State 1) with a constant force F applied to end of the cable. Assume that the cable does not slip on the pulley. The figure shows the system at State 1.

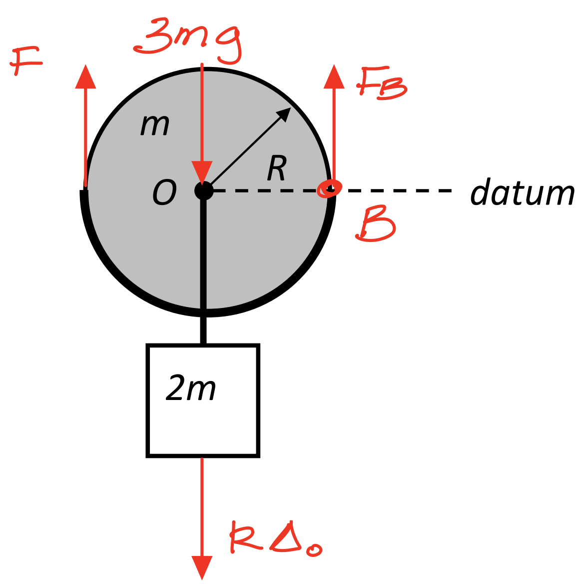


Find: Follow the solution steps below to determine the angular velocity ω of the pulley after point O has moved **up** a distance of d (State 2). Express your answer in terms of, at most m, h, d, R, k and the gravitational constant g. Write your answer as a vector.

Solution

Step 1: FBD

Use the spaces below to draw the FBD for the system consisting of both the pulley and the block. Your FBD must indicate the relevant coordinates.



Step 2 Kinetics:

Write down the work energy equation for the system based on the FBD in Step 1. Please use the provided datum. You may define and use any new unknown variables needed in Step 2 to express the following terms.

$T_1 =$	0 ; I.A.R
$V_1 =$	$-2mgh$
$U_{1\to2}^{NC} =$	$F\Delta_A$
$T_2 =$	$\frac{1}{2}I_B\omega^2 + \frac{1}{2}(2m)v_O^2$ [Note: B = IC of pulley]
$V_2 =$	$mgd + 2mg(d-h) + \frac{1}{2}k\Delta_O^2$

w/ $I_B = I_O + mR^2 = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$

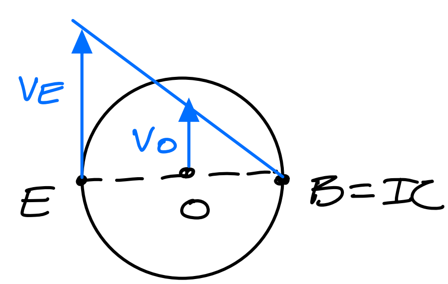
Step 3 Kinematics:

Write down the kinematics equations that are needed to solve this problem.

Since B = IC of pulley:

(1) $v_O = R\omega \Rightarrow \Delta_O = R\theta = d$

(2) $v_E = 2R\omega = 2v_O \Rightarrow \Delta_A = \Delta_E = 2d$



Problem 2 (continued)

***Step 4 Solve*:**

Using the equations from Steps 2 and 3 above, solve for the angular velocity ω of the pulley. Your answers should be in terms of, at most: m, h, d, R, k and g. Write your answer as a vector.

Substitute (1) and (2) into W/E equation:

$$\cancel{T_1} + V_1 + U_{1\to2}^{(nc)} = T_2 + V_2$$

$$-2m\cancel{g}h + F(2d) = \frac{1}{2}I_C\omega^2 + \frac{1}{2}(2m)(R\omega)^2 + mgd + 2mg(d-\cancel{h}) + \frac{1}{2}k\Delta_0^2$$

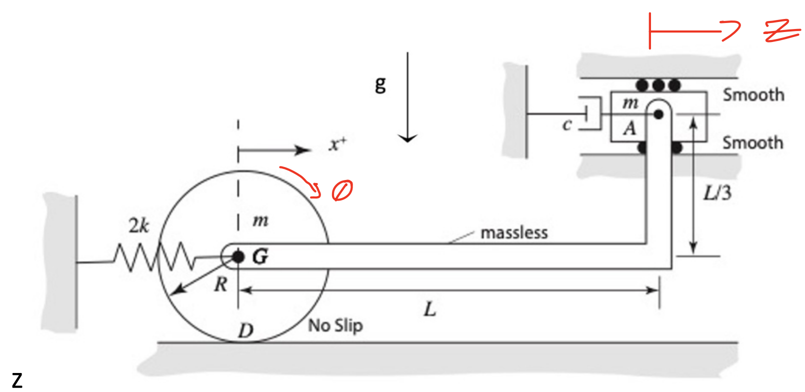
$$= \underbrace{\left[\frac{1}{2}\left(\frac{3}{2}mR^2\right) + mR^2\right]}_{\frac{7}{4}mR^2}\omega^2 + 3mgd + \frac{1}{2}kd^2$$

$$\hookrightarrow \vec{\omega} = -\left[\frac{(2F - 3mg)d - \frac{1}{2}kd^2}{\frac{7}{4}mR^2}\right]\hat{k} \longleftarrow \vec{\omega}$$

Problem 3 *(20 points)*

Given: Consider the system described below. It is comprised of a disk and a lumped mass, each with mass m and with the disk having a radius R. The disk and the lumped mass are interconnected via a rigid, massless link at points A and G, respectively. The relevant dimensions of the link are given on the figure. Attached to the disk at point G is a spring with a stiffness of $2k$. Additionally, a damper with a damping coefficient c is connected to the lumped mass m. The disk rolls without slip, while the lumped mass slides along a smooth surface. The mass moment of inertia of a disk about its center of gravity is $I_G = (1/2)mR^2$. The displacement of the center of the disk G is x.

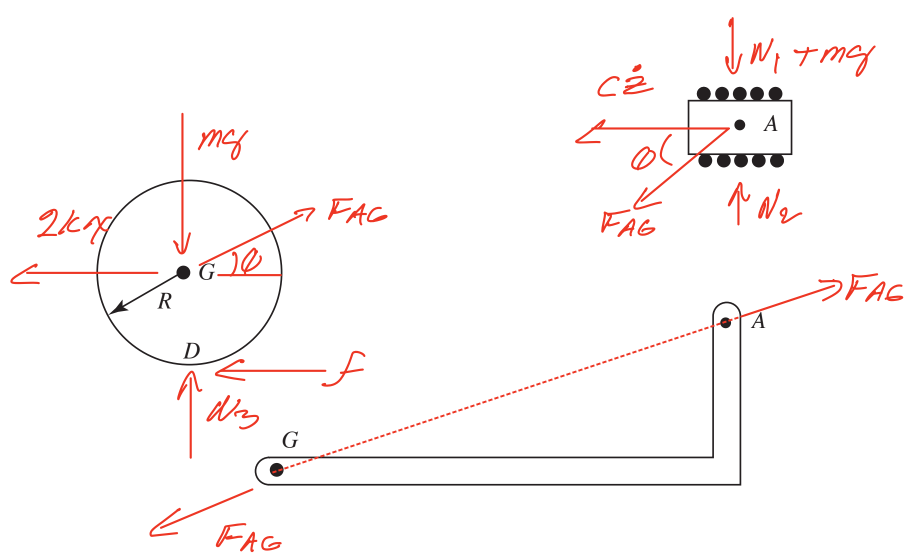
The known variables are k, c, m, L, R.



Find: The equation of motion of the system by completing the following steps.

Solution:

a) Draw FBDs of the disk, the massless link, and lumped mass on the figures provided.



Problem 3 *(continued)*

b) Write down the relevant Newton-Euler equations for each FBD.

Disk

$\xrightarrow{+} \Sigma F_x: \quad F_{AG}\cos\theta - f - 2kx = m\ddot{x}$ ①

$+\uparrow \Sigma F_y: \quad F_{AG}\sin\theta - mg + N = 0$

$\circlearrowright + \Sigma M_G: \quad +fR = I_G\ddot{\theta}, \quad I_G = \frac{1}{2}mR^2$ ② or $\circlearrowright + \Sigma M_C: -2kxR + F_{AG}\cos\theta R = I_C\ddot{\theta}$ ④

$I_C = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$

Block

$\xrightarrow{+} \Sigma F_z: \quad -c\dot{z} - F_{AG}\cos\theta = m\ddot{z}$ ③

$+\uparrow \Sigma F_y: \quad -N_1 + mg + N_2 - F_{AG}\sin\theta$

Required ①, ②, ③ or ③ and ④

c) Use kinematics to establish the relationship between the angular displacement, angular velocity, and angular acceleration of the disk and the translational displacement, translational velocity, and translational acceleration of the disk's center of gravity G, respectively.

assuming θ ↻ clockwise is positive

$x = R\theta$

$\dot{x} = R\dot{\theta}$

$\ddot{x} = R\ddot{\theta}$

d) State the relationship between the displacement, velocity and acceleration of the lumped mass and the disk's center of gravity.

$x = z$

$\dot{x} = \dot{z}$

$\ddot{x} = z$

Disk and block are rigidly connected

Problem 3 *(continued)*

e) Use the equations in parts (a), (b), (c), and (d) to derive the differential equation of motion (EOM) of the center of the disk G. This EOM should be written in terms of, at most, the known variables along with x, $\dot{x}$ and $\ddot{x}$. Put this equation in standard form with the coefficient of $\ddot{x}$ equal to one.

The easiest path to the EOM

$$\circlearrowleft^{+} \sum M_C: \; -2kxR + F_{AG}\cos\theta\, R = \tfrac{3}{2} mR^2\ddot{\theta}$$

$$\xrightarrow{+} \sum F_z: \; -c\dot{z} - F_{AG}\cos\theta = m\ddot{z}$$

divide $\sum M_C$ by R and add to $\sum F_z$

$$-2kx + \cancel{F_{AG}\cos\theta} - \cancel{F_{AG}\cos\theta} - c\cancel{\dot{z}}^{\,\dot{x}} = \tfrac{3}{2}m\cancel{R\ddot{\theta}}^{\,\ddot{x}} + m\cancel{\ddot{z}}^{\,\ddot{x}}$$

$$-2kx - c\dot{x} = \tfrac{3}{2}m\ddot{x} + m\ddot{x}$$

Rearrange

$$\underline{\tfrac{5}{2}m\ddot{x} + c\dot{x} + 2kx = 0}$$

f) Use the equation of motion in part (e) to determine the undamped natural frequency (ω_n) and damping ratio (ζ) for this system. These should be expressed in terms of, at most, the known variables.

$$\ddot{x} + \underbrace{\frac{2}{5}\frac{c}{m}}_{2\zeta\omega_n}\dot{x} + \underbrace{\frac{4k}{5m}}_{\omega_n^2}x = 0$$

Natural Frequency: $\underline{\omega_n = \sqrt{\frac{4}{5}k/m}}$

Damping Ratio: $2\zeta\omega_n = \frac{2}{5}c/m \longrightarrow \zeta\omega_n = c/5m$

$$\zeta = \frac{c}{5m\omega_n} = \frac{c}{5m\sqrt{\frac{4}{5}k/m}} = \frac{c}{\frac{10}{\sqrt{5}}\sqrt{mk}} = \frac{c}{\frac{\sqrt{5}\cdot 10\sqrt{mk}}{\cancel{\sqrt{5}\sqrt{5}} \to 5}}$$

$$\underline{\zeta = \frac{c}{2\sqrt{5}\sqrt{mk}}}$$

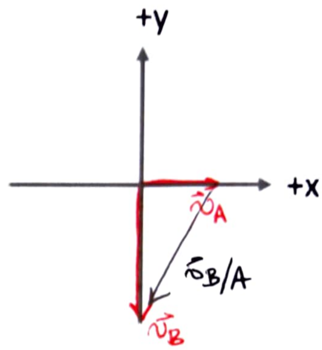
ME 274: Basic Mechanics II – Spring 2024 Name: SOLUTION
Final Exam – April 29

Problem 4 *(20 points)*
Part A (3 points)

Given: Two cars are approaching an intersection at constant speeds $\vec{v}_A = v\hat{i}$ and $\vec{v}_B = -2vj$ respectively as depicted in the figure. (No partial credit)

Find:

a) In the coordinate axes to the right, sketch the vectors $\vec{v}_A$ and $\vec{v}_B$.



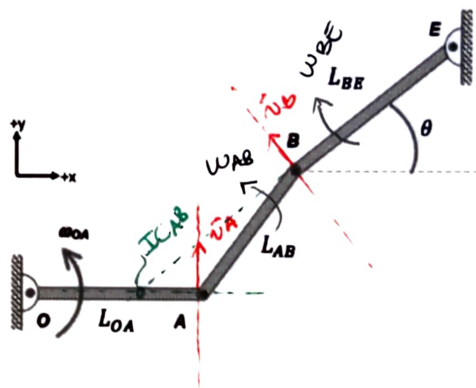
b) Write the velocity that car B will appear to have to an observer in car A in vector form, and sketch it in the coordinate axes to the right.

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = -2v\hat{j} - v\hat{i}$$

$$\vec{v}_{B/A} = -v\hat{i} - 2v\hat{j}$$

Part B (3 points)

Given: The mechanism shown in the figure consists of links OA, AB, and BE. Link OA is rotating counterclockwise at a rate ω_{OA}. (No partial credit)



Find:

a) Sketch the location of the instant center for link AB on the figure.

b) Is link AB rotating clockwise, counterclockwise, or not rotating at this instant? counterclockwise

c) Is link BE rotating clockwise, counterclockwise, or not rotating at this instant?

clockwise

d) Each of the three links has a mass of m, with the lengths of the links shown in the figure. Write expressions for the angular momentum of link OA about point O, $(\vec{H}_O)_{OA}$, and link BE about point E, $(\vec{H}_E)_{BE}$, at the instant shown. Use m, L_{OA}, L_{BE}, θ, ω_{OA}, and ω_{BE}.

$$(\vec{H}_O)_{OA} = I_O\vec{\omega}_{OA} = \left(I_G + m\left(\frac{L_{OA}}{2}\right)^2\right)\vec{\omega}_{OA} = \left(\frac{1}{12}mL_{OA}^2 + \frac{1}{4}mL_{OA}^2\right)\vec{\omega}_{OA}$$

$$(\vec{H}_O)_{OA} = \frac{1}{3}mL_{OA}^2\,\omega_{OA}\,\hat{k}$$

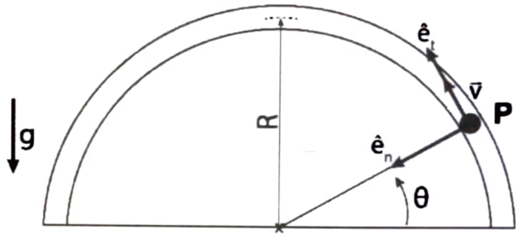
$$(\vec{H}_E)_{BE} = I_E\vec{\omega}_{BE} = \left(I_G + m\left(\frac{L_{BE}}{2}\right)^2\right)\vec{\omega}_{BE} = \left(\frac{1}{12}mL_{BE}^2 + \frac{1}{4}mL_{BE}^2\right)\vec{\omega}_{BE}$$

$$(\vec{H}_E)_{BE} = -\frac{1}{3}mL_{BE}^2\,\omega_{BE}\,\hat{k}$$

Name SOLUTION

Part C (2 points)

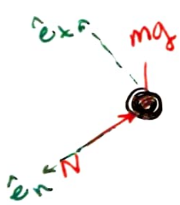
Given: A particle P moves along a smooth semicircular slot of radius $R = 1$ m in the **vertical plane** as depicted in the figure. At the instant shown, $\theta = 30°$ and the particle speed is $v = 2$ m/s. Use $g = 9.8$ m/s^2.



Find: Select the correct statement for the instant shown:

a) P is in contact with the inner surface of the slot
b) P is in contact with the outer surface of the slot
c) P is not in contact with any surface
d) There is not enough information to determine if P is in contact with a surface.

FBD



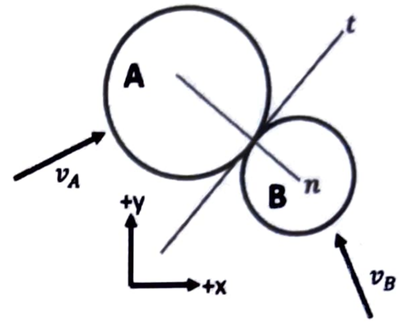
$$\swarrow \Sigma F_n = -N + mg\sin\theta = m\vec{a}_n = m\frac{v^2}{\rho}$$

$$N = m\left(g\sin\theta - \frac{v^2}{R}\right) > 0$$

ME 274: Basic Mechanics II – Spring 2024 **Name** SOLUTION
Final Exam – April 29

Part D(2 points)

Given: Smooth particles A and B impact each other as shown in the figure, with a coefficient of restitution $0 < e < 1$. All motion happens on a smooth **horizontal plane**.



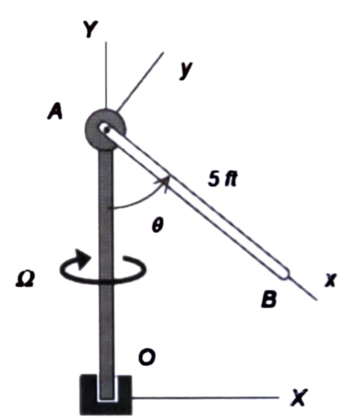
Find: Circle **ALL** the correct responses below. (No partial credit)

a) For ***system made up of A alone*** during impact:
   i) linear momentum in the x-direction is conserved
   ii) linear momentum in the y-direction is conserved
   iii) linear momentum in the n-direction is conserved
   iv) linear momentum in the t-direction is conserved
   v) energy is conserved

b) For ***system made of A and B*** during impact:
   i) linear momentum in the x-direction is conserved
   ii) linear momentum in the y-direction is conserved
   iii) linear momentum in the n-direction is conserved
   iv) linear momentum in the t-direction is conserved
   v) energy is conserved

Part E (4 points)

Given: The vertical shaft OA rotates about a fixed axis with a constant rate Ω. The arm AB is pinned to OA and is being raised at a constant rate $\dot{\theta}$. An observer and the xyz axes are attached to AB. The XYZ axes are stationary. (No partial credit)



Find:
a) What is the angular velocity vector for arm AB (xyz axes) when $\theta = 90°$?

$$\vec{\omega} = -\Omega\hat{J} + \dot{\theta}\hat{k} = -\Omega\hat{j} + \dot{\theta}\hat{k}$$

b) What is the angular acceleration vector for arm AB (xyz axes) when $\theta = 90°$?

$$\vec{\alpha} = \frac{d}{dt}(\vec{\omega}) = -\cancel{\dot{\Omega}\hat{J}}^{\,0} - \cancel{\Omega\dot{\hat{J}}}^{\,0} + \cancel{\ddot{\theta}\hat{k}}^{\,0} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}(\vec{\omega}\times\hat{k}) = \dot{\theta}(-\Omega\hat{J} + \dot{\theta}\hat{k})\times\hat{k}$$

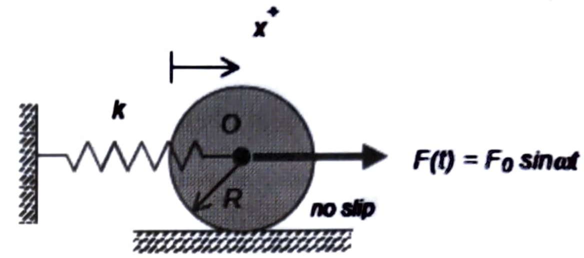
$$= \dot{\theta}(-\Omega\hat{j} + \dot{\theta}\hat{k})\times\hat{k} = -\Omega\dot{\theta}\hat{i}$$

ME 274: Basic Mechanics II – Spring 2024 Name SOLUTION

Final Exam – April 29

Part F (6 points)

Given: A homogeneous disk (mass m and radius R) rolls without slipping on a rough horizontal surface. A spring (stiffness k) is attached between the disk center O and ground, as shown in the figure. Let x describe the position of O such that the spring is unstretched when $x = 0$. (No partial credit)



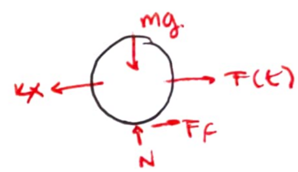
Find:

a) Is energy conserved in the system? Justify why or why not.

No, due to the externally applied force F(t)

b) What is the natural frequency of the system, ω_n?

FBD



Kinematics

$x = R\theta$

$\ddot{x} = R\ddot{\theta}$

$\circlearrowleft \sum M_C = Rkx - RF(t) = -I_C\ddot{\theta}$

$\cancel{R}kx - \cancel{R}F(t) = -\frac{3}{2}m\cancel{R}^2\left(\frac{\ddot{x}}{\cancel{R}}\right)$

$\frac{3}{2}m\ddot{x} + kx = F_0 \sin\omega t$

$\omega_n = \sqrt{\frac{2k}{3m}}$

$I_C = (I_G + mR^2) = \frac{3}{2}mR^2$

c) What is the amplitude of the steady state response for this system $X(\omega)$?

$\ddot{x} + \frac{2k}{3m}x = \frac{2F_0}{3m}\sin\omega t \rightarrow \ddot{x} + \omega_n^2 x = f_0 \sin\omega t, \quad f_0 = \frac{2F_0}{3m}$

$x_p(t) = A\sin\omega t + B\cos\omega t, \quad \ddot{x}_p(t) = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$[-A\omega^2\sin\omega t - B\omega^2\cos\omega t] + \omega_n^2[A\sin\omega t + B\cos\omega t] = f_0\sin\omega t$

$\sin\omega t: \quad A(\omega_n^2 - \omega^2) = f_0 \rightarrow A = f_0/(\omega_n^2 - \omega^2)$

$\cos\omega t: \quad B(\omega_n^2 - \omega^2) = 0 \rightarrow B = 0$

$\Rightarrow X(\omega) = \frac{f_0}{\omega_n^2 - \omega^2}$

d) At what forcing frequency $\omega > 0$ does the magnitude of the response $X(\omega)$ equal the magnitude of the response when $\omega = 0$? That is, when is $|X(\omega)| = |X(0)|$?

$|X(0)| = \frac{f_0}{\omega_n^2}, \quad |X(\omega)| = \frac{f_0}{\omega_n^2 - \omega^2}$

$\frac{\cancel{f_0}}{\omega_n^2} = \frac{\cancel{f_0}}{\omega_n^2 - \omega^2} \Rightarrow |\omega_n^2 - \omega^2| = \omega_n^2$

$\omega^2 - \omega_n^2 = \omega_n^2$

$\omega^2 = 2\omega_n^2$

$\omega = \sqrt{2}\,\omega_n$