Exam 3

Linear and Angular Impulse and Momentum, Central Impact, Newton/Euler
Rigid Body Work Energy

Last Name: ______________________ **First Name:** ______________________

Instructor (please circle your lecture time): (8:30am) Krousgrill (10:30am) Krousgrill

(11:30am) Gibert (1:30pm) Davies (2:30pm) Sotelo (4:30pm) Rojas

Instructions:

- You have 60 minutes to complete this exam.
- This is a closed-book, closed-notes exam.
- Use of mobile phones and other electronic communication devices is NOT permitted during the exam.
- You are allowed to use an approved calculator during the exam.

- Coordinate systems used must be clearly identified.
- Vectors must be clearly identified with proper vector notation.
- Units must be clearly stated as part of the answer.

- You must show all work to receive full credit.
- If the solution does not follow a logical thought process, it will be assumed to be in error.
- All the work you submit must be your own work.

- To ease scanning and uploading, only use the front sides of the following pages and start immediately after the problem statement.
- Put your name, problem number, and section number ON EACH PAGE.
- Read and sign (if you agree to abide to it) the Honor Statement on page (ii) of the exam
- Formula sheet is on page (iii) of the exam.

Problem	**Score**
1 (20 pts)	
2 (20 pts)	
3 (20 pts)	
Total	

Code of Honor for Students taking ME 274 Examinations

I understand that this is a CLOSED BOOK EXAM and I may only use my calculator to do simple arithmetic and trigonometric calculations.

During the examination, I will NOT USE crib sheets, course notes, text books, nor any other similar materials. I WILL NOT communicate with anyone else during the exam, nor will I accept help from anyone else. I WILL NOT use information on the internet or elsewhere to help me solve the exam problems.

All the work that I submit will be my own work.

I have read Code of Honor for Students taking ME274 and I agree to abide by this Code.

Signature: ______________________________

PRINT Last Name: ______________________________

PRINT First Name: ______________________________

Date: ______________________________

The Purdue Honor Code: *As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together. We are Purdue.*

Equation Sheet

$$\vec{v}_P = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = v_P\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

$$\vec{a}_P = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} = \dot{v}_P\hat{e}_t + \frac{v_P^2}{\rho}\hat{e}_n + 0\ \hat{e}_b = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$$

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$
$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$

$$\sum \vec{F} = m\vec{a}_P$$

$$T_1 + V_1 + U_{1\to 2}^{nc} = T_2 + V_2$$
where:

$T = \sum_i^N \frac{1}{2}m_i v_i^2$ is the kinetic energy of each particle and N is the number of particles,

$V = V_g + V_{sp}$ where $V_g = mgh,\ h$ is the signed distance above or below the datum,

$V_{sp} = \frac{1}{2}k(l - l_o)^2,$ and l, l_o are the deformed and original length of the spring, respectively, and

$U_{1\to 2}^{nc}$ is the work done by nonconservative forces.

$$m\vec{v}_{P_1} + \int_{t_1}^{t_2} \sum \vec{F} dt = m\vec{v}_{P_2} \qquad e = -\frac{v_{Bn2} - v_{An2}}{v_{Bn1} - v_{An1}}$$

$$\vec{H}_{O_1} + \int_{t_1}^{t_2} \sum \vec{M}_O dt = \vec{H}_{O_2} \qquad \vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$$

$$\sum \vec{F} = m\vec{a}_G \qquad \sum \vec{M}_A = I_A\vec{\alpha} + \vec{r}_{G/A} \times m\vec{a}_A$$

$$T_1 + V_1 + U_{1\to 2}^{nc} = T_2 + V_2 \qquad T = \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + m\vec{v}_A \bullet (\vec{\omega} \times \vec{r}_{G/A})$$

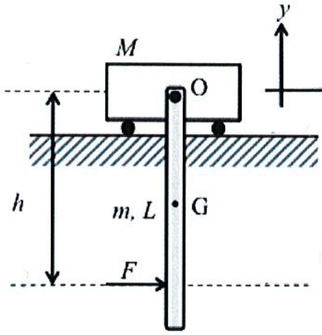
$$m\vec{v}_{G_1} + \int_{t_1}^{t_2} \sum \vec{F} dt = m\vec{v}_{G_2}$$

$$\vec{H}_{A_1} + \int_{t_1}^{t_2} \sum \vec{M}_A dt = \vec{H}_{A_2}$$

$\vec{H}_A = I_A\ \vec{\omega},$ for special cases of point A.

$$I_{G,disk} = \frac{1}{2}mr^2 \qquad I_{G,bar} = \frac{1}{12}mL^2 \qquad I_{G,plate} = \frac{1}{12}m\left(a^2 + b^2\right) \qquad I_{G,sphere} = \frac{2}{5}mr^2 \qquad I_{G,ring} = mr^2$$

Problem 1 (20 points):



Given:
A cart of mass M Kg is free to roll on a horizontal plane.

A thin slender rod of mass m Kg and length L meters is pinned to the cart at O. The rod hangs down in the vertical plane. G denotes the center of mass of the slender rod.

The cart and the rod are at rest when a horizontal force of F Newtons is applied to the rod h meters below O.

To Find/ To Do:

(a) Draw the free body diagrams of the thin slender rod and the cart.

(b) Write down the equation of motion of the block in the x-direction, and also the equation of motion of the rod in the x-direction. Use these equations to derive a *single* equation that relates F, M, m, the rate of change of speed of the cart (a_O), and the horizontal component of the acceleration of the center of mass of the rod (a_{Gx}).

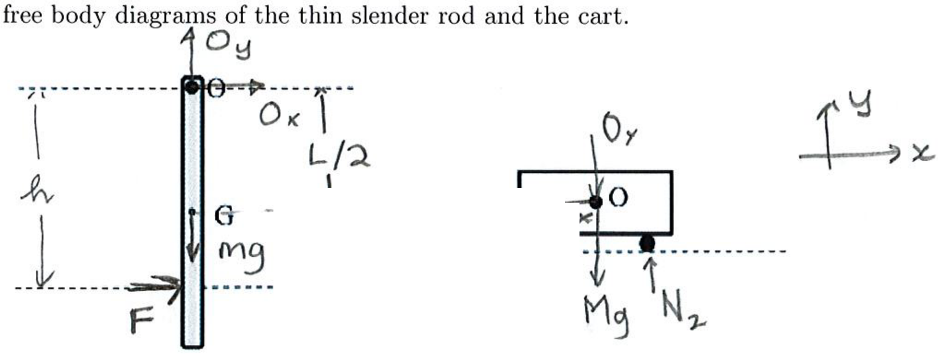
(c) Use Euler's equation for the rod taking moments about G, and the equation describing the motion of the cart, to derive a single equation relating F, m, M, L, h, a_0, and α, where $\vec{\alpha}$ is the angular acceleration of the rod. No other variables should appear in this equation.

(d) Use kinematics to derive a single expression relating a_{Gx}, a_O, and α.

(e) Determine $\vec{a}_O$, the acceleration of the cart at the instant the force F is applied. Your answer should only be in terms of all of, or a subset of, the known variables, which are: F, m, M, L, h, and g.

Solution:

(a) Draw the free body diagrams of the thin slender rod and the cart.



(b) Write down the equation of motioı tion, and also the equation of motion of the rod in the x-direction. Use these equations to derive a *single* equation that relates F, M, m, the rate of change of speed of the cart (a_O), and the horizontal component of the acceleration of the center of mass of the rod (a_{Gx}).

$$-O_x = M a_O$$
$$O_x + F = m a_{Gx}$$

Eliminate O_x

$$F = M a_o + m a_{Gx}$$ [redacted] Ans(b)

(c) Use Euler's equation for the rod taking moments about G, and the equation describing the motion of the cart, to derive a single equation relating F, m, M, L, h, a_O, and α, where $\vec{\alpha}$ is the angular acceleration of the rod. No other variables should appear in this equation.

$$-O_x L/2 + \left(h-\frac{L}{2}\right)F = I_G \alpha = \frac{1}{12} m L^2 \alpha$$

$$-O_x = M a_O$$

$$M a_O \frac{L}{2} + \left(h - \frac{L}{2}\right)F = \frac{1}{12} m L^2 \alpha$$ [redacted] Ans(c)

(d) Use kinematics to derive a single expression relating a_{Gx}, a_O, and α.

$$\bar{a}_G = \bar{a}_O + \bar{\alpha} \times \bar{r}_{G/O} - \omega^2 \bar{r}_{G/O}$$

$$a_{Gx}\hat{i} + a_{Gy}\hat{j} = a_o \hat{i} + \alpha \hat{k} \times \left(-\frac{L}{2}\hat{j}\right) - \omega^2\left(-\frac{L}{2}\hat{j}\right)$$

$\hat{i}$: $$a_{Gx} = a_o + \alpha \frac{L}{2}$$ [redacted] Ans(d)

(e) Determine the acceleration of the cart $\vec{a}_O$ at the instant the force is applied. Your answer should only be in terms of all of, or a subset of, the known variables. Known variables are: F, m, M, L, h, and g.

subst. 3 into 1

$$F = M a_o + m\left(a_o + \alpha \frac{L}{2}\right)$$ and rearrange

* $$F = (M+m) a_o + m\alpha \frac{L}{2}$$ [redacted]

$$\left(h - \frac{L}{2}\right)F = -\frac{ML}{2} a_o + \frac{1}{12} m L^2 \alpha$$ 2 rearranged

* $$\left(\frac{6h}{L} - 3\right)F = -3M a_o + \frac{1}{2} m L \alpha$$ [redacted] & scaled by $\frac{6}{L}$

Subtract 4 from 5 to eliminate α terms

$$\left(6\frac{h}{L} - 3 - 1\right)F = \left[-3M - (M+m)\right] a_o$$

Rearrange and put in vector form:

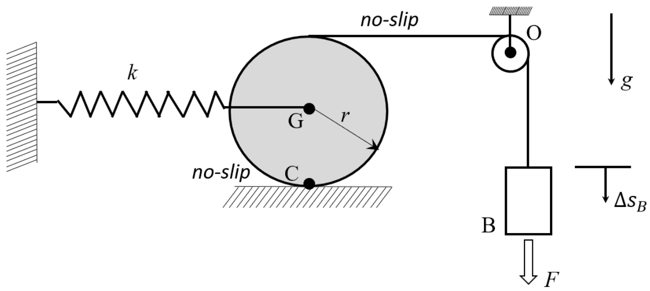
$$\vec{a}_o = \frac{(4 - 6h/L)F}{(4M+m)}\hat{i} \ m/s^2$$ ANS

OR

$$\text{or } \vec{a}_o = \frac{(4L - 6h)F}{(4M+m)L}\hat{i} \ m/s^2$$ ANS

Problem 2 (20 points)

Given: The rigid body system shown in the figure consists of a disk (mass m (kg), radius r (m)) and block, B (mass M (kg)). The disk rolls without slipping and is attached to an unstretched spring with stiffness k (N/m) at the center of mass, point G. A massless cable is wrapped over the top of the disk and connects to block B, which is hanging vertically over a fixed massless-frictionless pulley and pulled on by a force F (N) a distance Δs_B (m). Assume that the cable does not slip at the disk. The system is released from rest.



Find:

(a) Draw the free-body diagram of the **total system** (disk, cable, pulley, and block B). Clearly indicate your coordinate system **and** define your gravitational potential energy datum(s).

(b) Clearly write the Work-Energy equation and the individual terms associated including kinetic energies (T_i), potential energies (V_i), and non-conservative work ($U^{nc}_{1\to 2}$).

(c) Is energy for the total system conserved? (Provide a brief explanation)

(d) Determine the angular speed ω_2 of the disk.

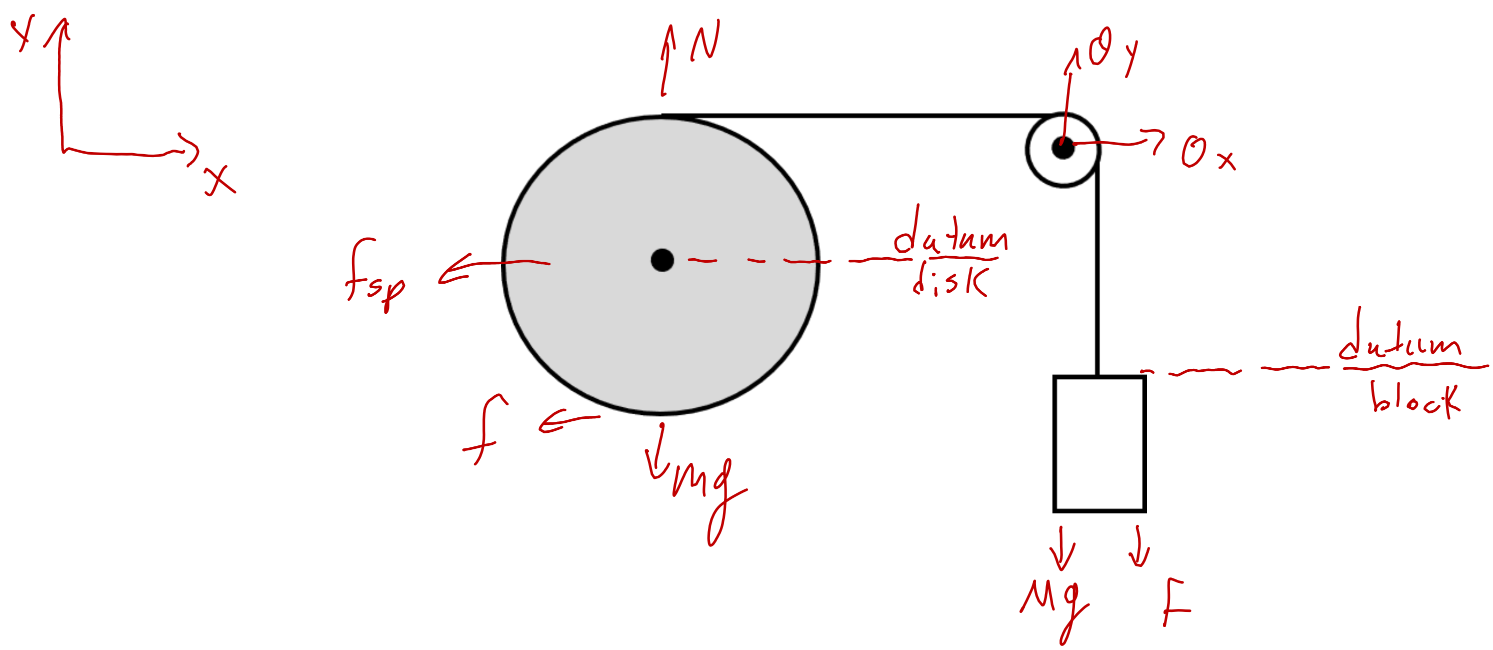
(e) Determine the speed of the center of mass of the disk G, v_{G2}.

(f) Determine the speed of block B, v_{B2}.

Units must be clearly stated as part of the answer (only at the very end) you do not have to carry units throughout the analysis. Note that your final answers should be in terms of (at most) the known variables: m, M, r, F, Δs_B, k, and gravity, g (m/s^2).

Solution:

(a) Draw a free-body diagram of the total system including a coordinate system and datum(s).



(b) Write out the Work-Energy equation in its full form (T,V,U) followed by filling out the contributions of energies associated with the system for each state (1,2).

$$\text{W-E: } \underline{T_1} + \underline{V_1} + \underline{U^{(nc)}_{1\to2}} = \underline{T_2} + \underline{V_2}$$

(i) $T_1 =$ 0 (released from rest)

(ii) $V_1 = V_{1spring} + V_{1gravity} =$ 0 (see datums)

(iii) $U^{nc}_{1\to2} =$ $+F\Delta s_B$ (applied external force)

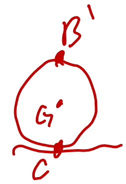
(iv) $T_2 =$ $\frac{1}{2} m v_{G2}^2 + \frac{1}{2} I_G \omega_2^2 + \frac{1}{2} M v_{B2}^2$ or $\frac{1}{2} I_C \omega_2^2 + \frac{1}{2} M v_{B2}^2$

(v) $V_2 = V_{2spring} + V_{2gravity} =$ $-Mg\Delta s_B + \frac{1}{2} k \Delta s_G^2 = -Mg\Delta s_B + \frac{1}{2} k \left(\frac{\Delta s_B}{2}\right)^2$

$\hookrightarrow \Delta s_G = \Delta s_B/2$

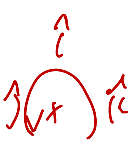
(c) Is energy for the total system conserved? Provide a very brief explanation.

NO. There is an externally applied force $\vec{F}$ doing work.



(d) Determine the angular speed of the disk, ω_2.

$$\cancel{T_1}^0 + \cancel{V_1}^0 + U^{nc}_{1\to2} = T_2 + V_2$$



$$\Rightarrow +F\Delta s_B = \frac{1}{2} m v_{G2}^2 + \frac{1}{2} I_G \omega_2^2 + \frac{1}{2} M v_{B2}^2 + \frac{1}{8} k \Delta s_B^2 - Mg\Delta s_B$$

$$\vec{v}_{G2} = \cancel{\vec{v}_C}^0 + \vec{\omega}_2 \times \vec{r}_{G/C} = \omega_2 \hat{k} \times r\hat{j} = -\omega_2 \hat{k} \times r\hat{j} = \omega_2 r \hat{i}$$

$$\vec{v}_{B2} = \vec{v}_{B'2} = \cancel{\vec{v}_C}^0 + \vec{\omega}_2 \times \vec{r}_{B'/C} = -\omega_2 \hat{k} \times 2r\hat{j} = \omega_2 2r \hat{i} = -v_{B2} \hat{j}$$

$$+F\Delta s_B = \frac{1}{2} m r^2 \omega_2^2 + \frac{1}{2} I_G \omega_2^2 + \frac{1}{2} M 4r^2 \omega_2^2 + \frac{1}{8} k \Delta s_B^2 - Mg\Delta SB$$

$\downarrow$ solve now for ω_2

(e) Determine the speed of the center of mass of the disk G, v_{G2}.

(d) cont.

$$\left(\frac{1}{2} m r^2 + \frac{1}{2} I_G + \frac{1}{2} M 4r^2\right)\omega_2^2 = Mg\Delta s_B - \frac{1}{8} k \Delta s_B^2 + F\Delta s_B$$

(f) Determine the speed of block B, v_{B2}.

$$\omega_2 = \sqrt{\frac{Mg\Delta s_B - \frac{1}{8} k\Delta s_B + F\Delta s_B}{\frac{1}{2} m r^2 + \frac{1}{4} m r^2 + 2Mr^2}}$$

(e) $\vec{V}_{G_2} = \vec{V}_C + \vec{\omega}_2 \times \vec{r}_{G/C} = \vec{0} - \omega_2 \hat{k} \times r\hat{j} = \omega_2 r \hat{i}$

$V_{G_2} = r\omega_2$

(f) $\vec{V}_{B_2} = \vec{V}_{B'2} = \vec{V}_C + \vec{\omega}_2 \times \vec{r}_{B'/C} = \vec{0} - \omega_2 \hat{k} \times 2r\hat{j} = 2r\omega_2 \hat{i}$

$\downarrow$

$V_{B_2} = 2r\omega_2$

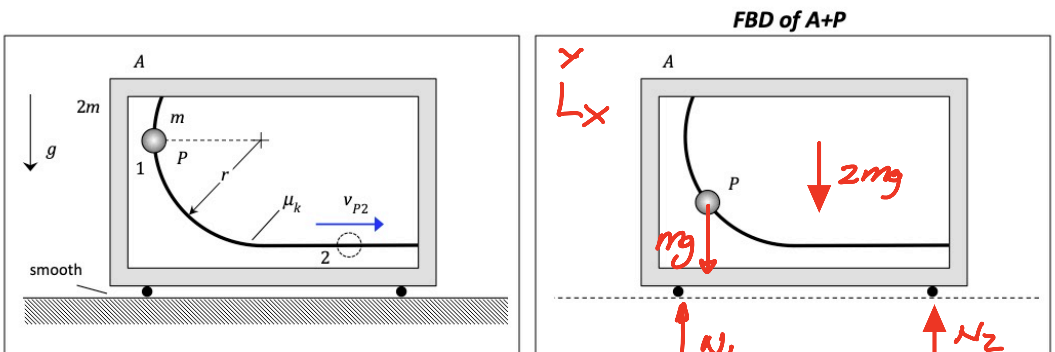
$\vec{V}_{B_2} = -2r\omega_2 \hat{j}$

 Last Name: SOLUTION

NOTE: *You are asked to provide justification (written and/or through equations) for your answers here in Problem 3. A correct response, alone, will receive only partial credit. Your work will be graded.*

Problem 3A(8 points):

Given: Particle P, having a mass of m, is constrained to move along a rough guide that is attached to box A (with A having a mass of $2m$). The coefficient of kinetic friction between P and the guide is $\mu_k = 0.6$. Box A is able to move along a smooth, horizontal surface. The system is released from rest with P at Position 1 shown. At Position 2, P is moving along a segment of the guide that is horizontal. Let v_{A2} and v_{P2} be the speeds of A and P, respectively, at Position 2.



PART 3A.1: Using the figure above, draw the free body diagram (FBD) of the combined system of A+P.

PART 3A.2: Circle the correct answer below regarding the direction of movement of A at position 2:

(a) A is moving to the right.
(b) A is stationary
$\boxed{\text{(c) A is moving to the left.}}$
(d) More information is needed for answering this question.

PART 3A.3: Circle the correct answer below regarding the speeds of A and P at position 2:

(a) $v_{A2} > v_{P2}$
(b) $v_{A2} = v_{P2}$
$\boxed{\text{(c) } v_{A2} < v_{P2}}$
(d) More information is needed for answering this question.

PART 3A.4: Provide justifications for your answers above in PARTS 3A.2 and 3A.3.

$$\Sigma F_x = 0 \Rightarrow m \cancel{v_{P1x}}^{0} + 2m \cancel{v_{A1}}^{0} = m v_{P2x} + 2m v_{A2} = 0$$

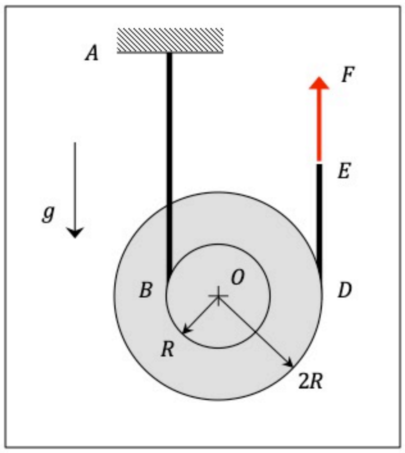
$$\hookrightarrow v_{A2} = -\tfrac{1}{2} v_{P2x}$$

∴ A moves to the LEFT with a speed of $v_{A2} = \tfrac{1}{2} v_{P2}$

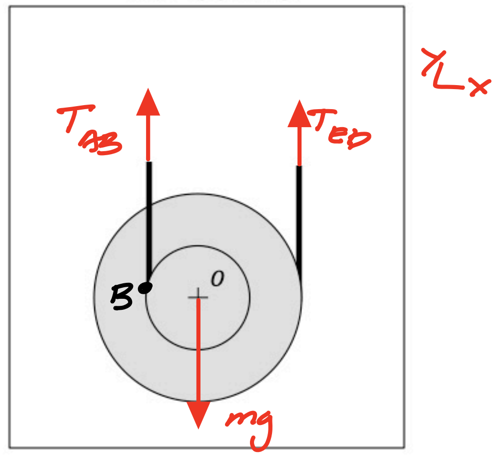
Last Name: SOLUTION

Problem 3B (6 points):

Given: Consider the stepped pulley shown with inner and outer radii of R and $2R$, respectively, and with a mass of m. Cable AB is wrapped around the inner radius, while a second cable ED is wrapped around the outer radius, as shown in the figure below. The center of the pulley, O, is known to be accelerating in the DOWNWARD direction. Let T_{AB} and T_{ED} represent the tensions in cable AB and ED.



FBD of pulley



PART 3B.1: Using the figure above, draw the free body diagram (FBD) of the stepped pulley.

PART 3B.2: Circle the correct answer below regarding the size of T_{ED}:

(a) $T_{ED} > 2mg$
(b) $T_{ED} = 2mg$
(c) $2mg > T_{ED} > mg$
(d) $T_{ED} = mg$
(e) $\boxed{T_{ED} < mg}$
(f) More information is needed for answering this question.

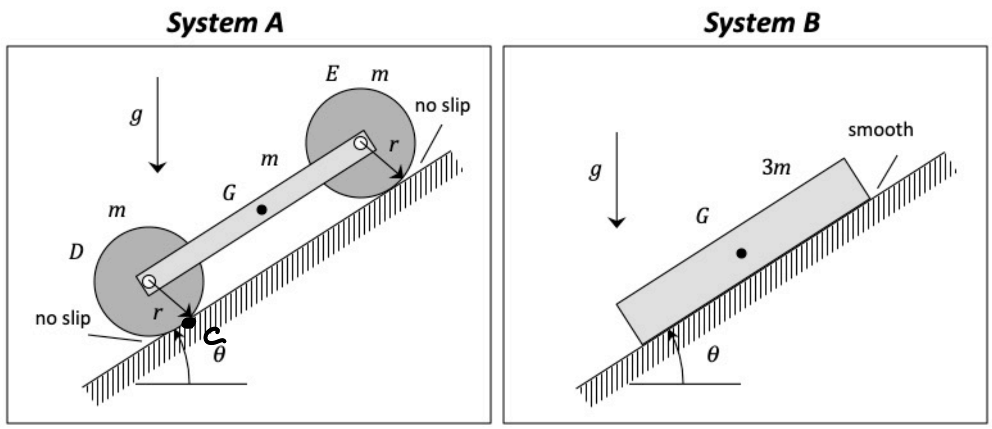
PART 3B.3: Provide justification for your answer above in PART 3B.2.

$$\Sigma M_B = -mgR + T_{ED}(2R) = I_B\alpha \Rightarrow 2T_{ED} = mgR + I_B\overset{<0}{\alpha}$$
$$\hookrightarrow T_{ED} < \frac{mg}{2}$$

OR $$\Sigma F_y = T_{AB} + T_{ED} - mg = ma_O \Rightarrow T_{ED} = mg + m\overset{<0}{a_O} - T_{AB}$$
$$\hookrightarrow T_{ED} < mg$$

Problem 3C (6 points):

Given: System A shown below is made up of a homogeneous rigid bar (having a mass of m) connecting the centers of two homogeneous wheels, with each wheel having a mass of m and radius r. The wheels roll without slipping on the incline as the system moves. System B is a homogeneous bar of mass $3m$ that is able to slide on a smooth incline, with this incline being at the same angle θ as in System A. Both systems are released from rest, with the centers of mass G for both bars at the same height. Let v_{A2} and v_{B2} be the speeds of the bars' centers of mass for Systems A and B, respectively, after these centers of mass have moved along the same distance d on the incline.



PART 3C.1: Write down expressions for the kinetic energy for Systems A and B in terms of v_{A2} and v_{B2}, along with m and possibly r.

$$T_A = \frac{1}{2} m v_A^2 + 2\left(\frac{1}{2} I_C \omega^2\right) \; ; \quad I_C = \frac{3}{2} m r^2 \;\&\; \omega = \frac{v_A}{r}$$

$$= \frac{1}{2} m v_A^2 + \frac{3}{2} m v_A^2 = 2 m v_A^2$$

$$T_B = \frac{1}{2} 3m v_B^2$$

PART 3C.2: Circle the correct response below regarding the speeds v_{A2} and v_{B2}:

(a) $v_{A2} > v_{B2}$
(b) $v_{A2} = v_{B2}$
$\boxed{\text{(c) } v_{A2} < v_{B2}}$
(d) More information is needed to answer this.

PART 3C.3: Provide justification for your answer above in PART 3C.2.

$$\text{Since } V_A = V_B \Rightarrow T_A = T_B \Rightarrow$$

$$2 m v_A^2 = \frac{3}{2} m v_B^2 \Rightarrow v_A = \sqrt{\frac{3}{4}}\, v_B$$