Lecture 7

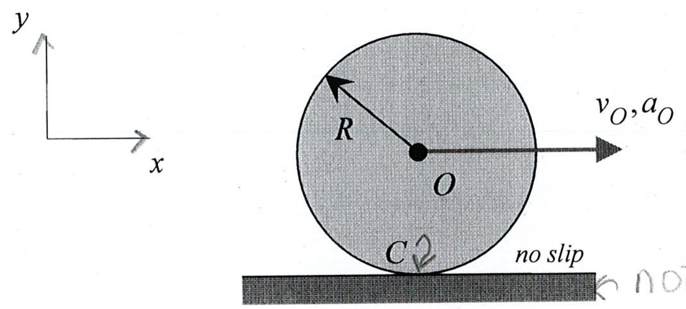
Rolling Without Slipping

Consider the wheel shown below that rolls along a rough, stationary horizontal surface with the center of the wheel O having a velocity and acceleration of v_O and a_O, respectively. As the wheel moves, it is assumed that sufficient friction acts between the wheel and the fixed horizontal surface that the contact point C does not slip. The consequences of the "no slip" condition at C are:

$$v_{cx} = 0$$

$$a_{cx} = 0$$

(If either of the above is not true, then point C "slips" as the wheel moves.)



path of O a straight line

$\vec{v}_O = v_O\hat{i}$

$\vec{a} = a_O\hat{i}$

not moving

We will encounter many problems throughout the course that involve the rolling without slipping of a body on a stationary surface. Although this concept is defined through a simple set of equations, the consequences of rolling without slipping on the velocity and acceleration of other points on the body can become quite complicated. We will see this through a number of examples.

CHALLENGE QUESTION: If C is a no-slip point, what are the y-components for the velocity and acceleration of C?

ANSWER: Since O moves on a straight, horizontal path, the y-components of the velocity and acceleration of O are zero. From this and the no-slip condition above for C, we can write:

no slip

$$\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O}$$

$$v_{Cx}\hat{i} + v_{Cy}\hat{j} = v_O\hat{i} + (\omega\hat{k}) \times (-R\hat{j}) = (v_O + R\omega)\hat{i} \quad \Rightarrow \quad \boxed{v_{Cy} = 0}$$

O (no $\hat{j}$ component on R.H.S.)

and $v_O = -R\omega$ (negative (clockwise rotation))

and

$$\vec{a}_C = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2 \vec{r}_{C/O}$$

$$a_{Cx}\hat{i} + a_{Cy}\hat{j} = a_O\hat{i} + (\alpha\hat{k}) \times (-R\hat{j}) - \omega^2(-R\hat{j})$$

$$= (a_O + R\alpha)\hat{i} + R\omega^2\hat{j} \quad \Rightarrow \quad a_{Cy} = R\omega^2$$

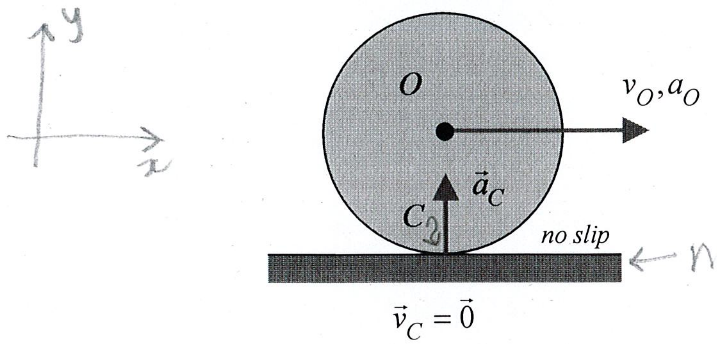
O

From this we see that: $\vec{v}_c = \vec{0}$ and $\vec{a}_C = R\omega^2\hat{j} \neq \vec{0}$.

Also $a_O + R\alpha = 0$ [No $\hat{i}$ component on L.H.S.]

$a_O = -R\alpha$

This says that the velocity for a no-slip point is zero; however, the acceleration of that point is NOT zero., it has a non-zero y component.

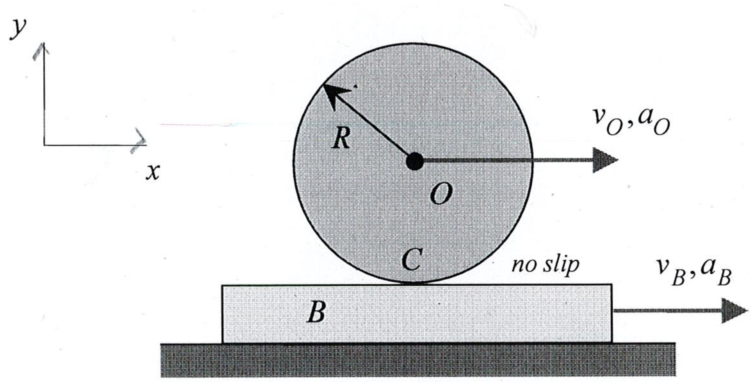


O moves in a straight line

← not moving

$a_{cx} = 0$ $a_{cy} \neq 0$ [C does not move on a straight line]

Consider another situation where here the wheel rolls without slipping on a second body B that is itself translating in the x-direction.



moving

No slip for this situation is described by the following:

$v_{cx} = v_B$ ✓ $v_{cy} = 0$

$a_{cx} = a_B$ ✓ $a_{cy} \neq 0$

Following the same type of analysis shown above, we can show that the no-slip conditions produces: $v_{Cy} = 0$ and $a_{Cy} \neq 0$.

$$\bar{v}_C = \bar{v}_O + \bar{\omega} \times \bar{r}_{C/O} =$$

$$v_B \hat{i} + v_{cy} \hat{j} = v_O \hat{i} + \omega \hat{k} \times (-R\hat{j})$$

$$= v_O \hat{i} + \omega R \hat{i} = (v_O + \omega R)\hat{i}$$

$\hat{i}$: $v_B = v_O + \omega R \longrightarrow v_O = v_B - \omega R$

$\hat{j}$: $v_{cy} = 0$

Example 2.A.4

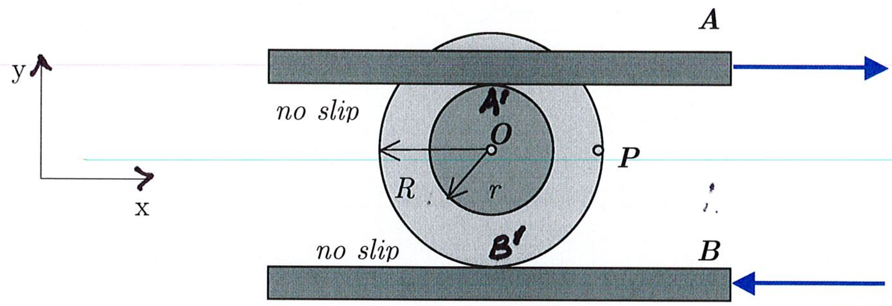
Given: Rack A moves to the right with a speed of v_A and an acceleration of a_A. Rack B moves to the left with a constant speed of v_B. Assume $v_A = 0.8$ m/s, $a_A = 2$ m/s^2, $v_B = 0.6$ m/s, $r = 0.1$ m and $R = 0.16$ m.

Find: Determine:

(a) The velocity of point P on the outer rim of the gear; and

(b) The acceleration of point P on the outer rim of the gear.

A′ and B′ are on the gear



$\bar{\omega} = \omega\hat{k}$, $\vec{\alpha} = \alpha\hat{k}$

Expect ω to be negative, also α.

Strategy

- Relate velocity at A′ to velocity at B′ to determine ω.
- Relate acceleration at A′ to acceleration at B′, and then use $\hat{i}$ components to determine α
- $\vec{v}_P = \vec{v}_{B'} + \bar{\omega} \times \vec{r}_{P/B'} = -v_B\hat{i} + \omega\hat{k} \times (R\hat{i} + R\hat{j})$
- Relate $\bar{a}_O$ to $\bar{a}_{B'}$, use $\hat{i}$ components to find $\bar{a}_O = a_O\hat{i}$

  O moves in a straight line
- Relate $\vec{a}_P$ to $\bar{a}_O$ to find $\vec{a}_P$

  $\vec{a}_P = \vec{a}_O + \bar{\alpha} \times \bar{r}_{P/O} - \omega^2 \vec{r}_{P/O}$

First find ω and α (A' from B')

$$\bar{v}_{A'} = \bar{v}_{B'} + \bar{\omega} \times (R+r)\hat{j}$$

$$v_A i = -v_B \hat{i} + \omega\hat{k} \times (R+r)\hat{j}$$

$$v_A = -v_B - (R+r)\omega$$

$$\omega = -\frac{(v_A + v_B)}{(R+r)} \text{ rad/s}$$

$$\vec{a}_{A'} = \vec{a}_{B'} + \vec{\alpha} \times \vec{r}_{A'/B'} - \omega^2 \vec{r}_{A'/B'}$$

$$\underline{a_A}\hat{i} + a_{A'_y}\hat{j} = +\underline{a_B}\hat{i} + a_{B'_y}\hat{j} + \underline{\alpha\hat{k} \times (R+r)\hat{j}} - \omega^2 (R+r)\hat{j}$$

i: $a_A = \cancel{a_B}^{\,0} - (R+r)\alpha$

constant v_B 0

$$\alpha = -\frac{a_A}{(R+r)} \text{ rad/s}^2$$

clockwise if a_A is positive

Let's find $\bar{v}_p$ and $\bar{a}_p$

$$\vec{v}_P = \vec{v}_{B'} + \bar{\omega} \times \bar{r}_{P/B'} =$$

$$\bar{v}_P = -v_B \hat{i} + \omega \hat{k} \times (R\hat{i} + R\hat{j})$$

$$\bar{v}_P = -(v_B + R\omega)\hat{i} + R\omega \hat{j} = \ldots \text{ m/s}$$

m/s

$$\vec{a}_O = \bar{a}_{B'} + \bar{\alpha} \times \bar{r}_{O/B'} - \omega^2 \bar{r}_{O/B'}$$

$$a_O \hat{i} = \underset{\swarrow}{a_{Bx}} \hat{i} + \hat{a}_{B'_y} \hat{j} + \underbrace{\alpha \hat{k} \times (R\hat{j})}_{-\alpha R \hat{i}} - \omega^2 R \hat{j}$$

straight line path

0 Given

$$\hat{i}: \quad a_O = -\alpha R$$

$$\vec{a}_O = -\alpha R \hat{i} \text{ m/s}^2$$

$$\vec{a}_p = \vec{a}_O + \vec{\alpha} \times \bar{r}_{P/O} - \omega^2 \bar{r}_{P/O}$$

$$= -\alpha R \hat{i} + \alpha \hat{k} \times R\hat{i} - \omega^2 (R\hat{i})$$

$$\bar{a}_p = (-\alpha R - \omega^2 R)\hat{i} + \alpha R \hat{j}$$

$$\bar{a}_p = -(\alpha R + \omega^2 R)\hat{i} + \alpha R \hat{j} \quad \text{m/s}^2$$

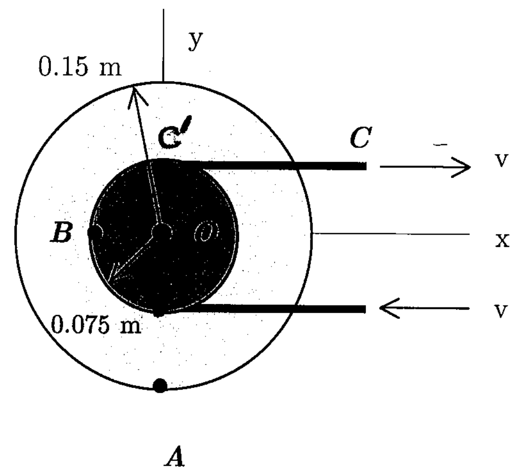
Example 2.A.2

Given: The belt-driven pulley and attached disk are rotating about a shaft passing through O. At a certain instant, the speed v of the belt is known to be 1.5 m/s and the MAGNITUDE of the acceleration of point A on the disk is 75 m/s^2. Assume that the belt does not slip on the disk.

Find: For this instant:
(a) Determine the angular acceleration of the pulley and disk;
(b) Determine the acceleration vector of point B on the disk; and
(c) Determine the acceleration of point C on the belt.

NB. We need more information to determine sign of α

O is a fixed point, $\therefore \vec{v}_O = \vec{0}$ and $\vec{a}_O = \vec{0}$



<u>Strategy</u>

(1) Use $\vec{a}_{C'} = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C'/O} - \omega^2 \vec{r}_{C/O}$
to show that you need to find ω before you can solve for α.

(2) Use $\vec{v}_{C'} = \vec{v}_O + \vec{\omega} \times \vec{r}_{C'/O}$ to find ω
and then solve for $|\alpha|$.

(3) $\vec{a}_B = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha} \times \vec{r}_{P/O} - \omega^2 \vec{r}_{P/O}$

assume $\alpha\hat{k}$, α positive

(4) $\vec{a}_C = a_{C'_x}\hat{i}$

$\vec{a}_{C'} = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C'/O} - \omega^2 \vec{r}_{C/O}$

Problem 2A2

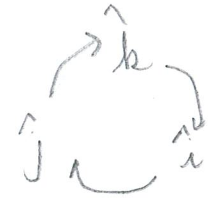
what we know:

clockwise rotation.

no-slip $\rightarrow$ $\vec{v}_{C'} = v\,\hat{i}$

$\vec{v}_D = -v\,\hat{i}$

$\vec{v}_0 = \vec{0}$ fixed point



At this instant $v = 1.5\,m/s$

$|\vec{a}_A| = 75\ m/s^2$

(a) Find $\vec{\alpha}$

$$\bar{a}_A = \bar{a}_0 + \vec{\alpha} \times \bar{r}_{A/0} - \omega^2 \bar{r}_{A/0}$$

$$a_{Ax}\hat{i} + a_{Ay}\hat{j} = \vec{0} + \alpha\hat{k} \times (-0.15\hat{j}) - \omega^2(-0.15\hat{j})$$

$$= 0.15\alpha\,\hat{i} + 0.15\omega^2\,\hat{j}$$

$$\sqrt{a_{Ax}^2 + a_{Ay}^2} = 75 = \sqrt{(0.15\underset{?}{\alpha})^2 + (0.15\underset{?}{\omega^2})^2}$$

Find $\omega \rightarrow \bar{v}_{C'} = \cancel{\bar{v}_0}^{\,0} + \bar{\omega} \times \bar{r}_{C'/0} = \omega\hat{k} \times (+0.075\hat{j})$

fixed point

$$v\hat{i} = -0.075\,\omega\,\hat{i} \rightarrow \omega = \frac{v}{-0.075} = -\frac{1.5}{0.075}$$

clockwise

$$\therefore\ \omega = -\frac{\cancel{1500}\ 500}{\cancel{75}\ 25} = -20\ rad/s$$

clockwise

$$|\alpha| = \sqrt{\frac{75^2 - \{0.15(-20)^2\}^2}{(0.15)^2}}$$

Note we do not know the sign of α (Need more info. Let's assume $\vec{\alpha} = \alpha\hat{k}$

$$\vec{\alpha} = \underset{?\uparrow}{\pm} |\alpha|\,\hat{k}\ \ rad/s^2$$

(b) $\vec{a}_B = \cancel{\vec{a}_0}^{\,0} + (\alpha\hat{k}) \times \vec{r}_{B/0} - \omega^2 \vec{r}_{B/0}$

(b) - continued

$$\vec{a}_B = \alpha\hat{k} \times (-0.075\hat{i}) - \omega^2(-0.075\hat{i})$$

$$\bar{a}_B = -0.075\alpha\hat{j} + 0.075\omega^2\hat{i} \quad m/s^2$$

$= \ldots\ldots$

(c) On the belt acceleration is to the right and is equal to x component of acceleration of point D on the disk.

$$\vec{a}_{c'} = \cancel{\vec{a}_o} + \alpha\hat{k} \times \vec{r}_{c'/o} - \omega^2\bar{r}_{c'/o}$$

fixed point

$$= \alpha\hat{k} \times 0.075\hat{j} - \omega^2\, 0.175\hat{j}$$

$$= -0.075\alpha\hat{i} - \omega^2\, 0.175\hat{j}$$

$$\bar{a}_c = a_{c'_x}\hat{i} = -0.075\alpha\hat{i} \quad m/s^2$$