Kinematics for Rigid Body (2D)

$\begin{cases} \text{motion of one point} \\ \text{rotation of rigid body} \end{cases} \Rightarrow$ motion of all points on rigid body

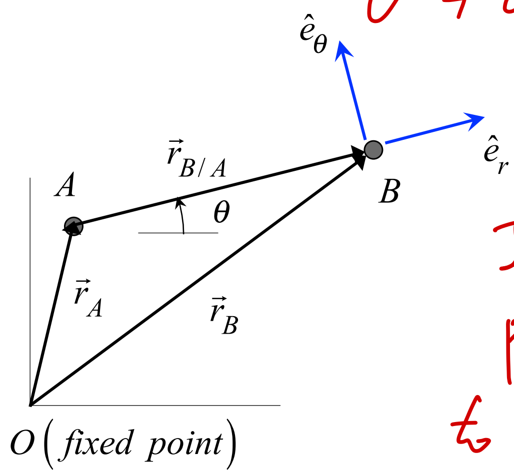
Background

In the last lecture, we saw that the position of a point B relative to a second point A can be written as:

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

Rigid body: $|\vec{r}_{B/A}| = r =$ Const

$\theta \neq$ Const $\rightarrow$ rotate



$\downarrow$

Intuitive to use polar coordinates to express $\vec{r}_{B/A}$

Using the description of $\vec{r}_{B/A}$ in terms of the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ shown, we can write:

$$\vec{r}_{B/A} = r\ \hat{e}_r = \vec{r}_B - \vec{r}_A$$

where $r = \left|\vec{r}_{B/A}\right|$ is the distance from A to B and $\hat{e}_r$ is aligned with $\vec{r}_{B/A}$ as shown above. Differentiation of this equation with respect to time and using our earlier results from polar kinematics gives:

$$\vec{v}_{B/A} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \vec{v}_B - \vec{v}_A$$

$$\vec{a}_{B/A} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = \vec{a}_B - \vec{a}_A$$

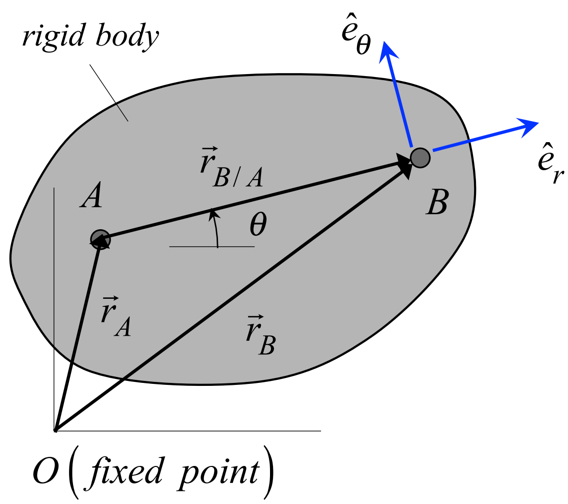
These equations are valid for planar motion of two general points A and B.

Objectives

The goals of this lecture are to: (i) develop and use the velocity and acceleration kinematic equations for the planar motion of a rigid body, and (ii) use the concept of *instant centers* to study the velocity of points on a rigid body moving in a plane.

A. Development of Kinematic Equations for Planar Rigid Body Motion

A *rigid body* is an object for which the distance between any two points on the object remains fixed regardless of the motion of the object. Consider points A and B that lie on the same rigid body with A and B moving in the same plane (or in parallel planes) for all time (*planar motion*). Since they are fixed on the same rigid body, we see that $\left|\vec{r}_{B/A}\right| = r = constant$. Although the distance between points A and B is constant, the vector's orientation angle θ changes with time as the body rotates. Therefore, the vector $\vec{r}_{B/A}$ is not constant.



$\dot{\theta} \neq 0$

Velocity and Acceleration Equations

Since A and B lie on the same rigid body, then $r = \left|\,\vec{r}_{B/A}\right|$ = CONSTANT and $\dot{r} = 0$ for all time. With this, the preceding kinematic equations reduce to:

we want to derive equations that apply to any coordinates

$$\begin{aligned}
\vec{v}_{B/A} &= \vec{v}_B - \vec{v}_A \\
&= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \\
&= r\dot{\theta}\hat{e}_\theta \qquad ; \quad \dot{r} = 0 \\
&= r\dot{\theta}\left(\hat{k} \times \hat{e}_r\right) \qquad ; \quad \hat{e}_\theta = \hat{k} \times \hat{e}_r \quad (by\ the\ RIGHT\ HAND\ RULE) \\
&= \left(\dot{\theta}\hat{k}\right) \times \left(r\hat{e}_r\right) \\
&= \vec{\omega} \times \vec{r}_{B/A}
\end{aligned}$$

for polar coord. only

→ write in terms of vector operations

→ for any coord. → also for 3D motion

where $\vec{\omega} = \dot{\theta}\hat{k}$ is the "angular velocity" of the rigid body.

Imposing the $|\vec{r}_{B/A}| = r = constant$ constraint equation on the relative acceleration equation gives:

$$
\begin{aligned}
\vec{a}_{B/A} &= \vec{a}_B - \vec{a}_A \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \\
&= \boxed{\left(-r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta}\right)\hat{e}_\theta} && ;\quad \dot{r} = \ddot{r} = 0 \\
&= \left(r\dot{\theta}^2\right)\left[\hat{k} \times \left(\hat{k} \times \hat{e}_r\right)\right] + \left(r\ddot{\theta}\right)\left(\hat{k} \times \hat{e}_r\right) && ;\quad \hat{e}_r = -\hat{k} \times \left(\hat{k} \times \hat{e}_r\right) \\
&= \dot{\theta}\hat{k} \times \left[\left(\dot{\theta}\hat{k}\right) \times (r\hat{e}_r)\right] + \left(\ddot{\theta}\hat{k}\right) \times (r\hat{e}_r) \\
&= \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}
\end{aligned}
$$

for polar coord.

for any coord. 3D motion

$\vec{w} = \dot{\theta}\hat{k}$

$\vec{\alpha} = \ddot{\theta}\hat{k}$

$\vec{r} = r\hat{e}_r$

$\hat{e}_\theta = \hat{k} \times \hat{e}_r$

$= -\omega^2 \vec{r}_{B/A} + \vec{\alpha} \times \vec{r}_{B/A}$ → for any coord, 2D motion

where $\vec{\alpha} = \ddot{\theta}\hat{k}$ is the "angular acceleration" of the rigid body.

Therefore, the velocity and acceleration vectors for point B referenced to point A (on the same rigid body) are given by:

$$
\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \\
\vec{a}_B &= \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}
\end{aligned}
$$

→ rotation, relative motion

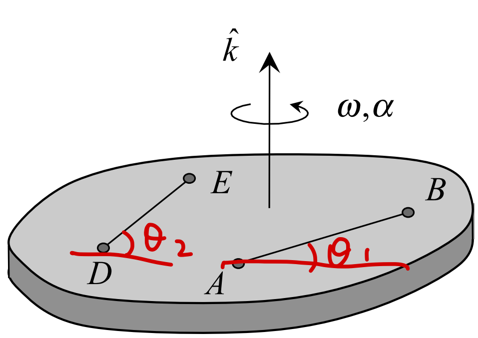
translation

Recall that the relative position vector $\vec{r}_{B/A}$ extends FROM point A TO point B. In our derivation, we wrote this vector in terms of its polar coordinates. In truth, this vector can be written in terms of any set of coordinates. In our problem solving, we will commonly write $\vec{r}_{B/A}$ in terms of a set of Cartesian coordinates, using unit vectors $\hat{\imath}$ and $\hat{\jmath}$. In doing so, we need to use the right hand rule in orienting these unit vectors relative to the $\hat{k}$ vector that appears in $\vec{\omega}$ and $\vec{\alpha}$; that is, we need to ensure that $\hat{k} = \hat{\imath} \times \hat{\jmath}$.

Chapter 2

Angular Velocity and Angular Acceleration

In the preceding development, we defined the angular velocity and angular acceleration of the body as $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$ and $\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$, respectively.



$\theta_1 = \theta_2 +$ const

$\Rightarrow w = \dot{\theta}_1 = \dot{\theta}_2$, $\alpha = \ddot{\theta}_1 = \ddot{\theta}_2$

These two vectors are defined with the direction of the unit vector $\hat{k}$ with this vector representing the axis of rotation for the body. The positive sense of the vectors are defined by the right hand rule (curl your fingers around the $\hat{k}$ axis and your fingers define the positive sense). As shown in the derivation, the scalars $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$ are time derivatives of the angle of the vector $\vec{r}_{B/A}$ from A to B. Note that the angle $\vec{r}_{E/D}$ for another pair of points D and E on the body differs from the angle θ only by a constant angle (since all points are attached to the same rigid body). Therefore, we conclude that the angular velocity and angular acceleration vectors, $\vec{\omega}$ and $\vec{\alpha}$, describe the same angular motion for any pair of points on the body. In other words, $\vec{\omega}$ and $\vec{\alpha}$ describe the angular motion of the BODY and are the same for any pair of points on the rigid body.

Discussion - Rigid Body Kinematics Equations

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \ \vec{\alpha} \times \vec{r}_{B/A} + \ \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

- The above two equations relate the kinematics of any two points A and B on the same rigid body while the body executes planar motion.

- The vectors $\vec{\omega} = \omega\hat{k}$ and $\vec{\alpha} = \alpha\hat{k}$ are the *angular velocity* and *angular acceleration* of the rigid body, respectively. These are properties of the motion of the rigid body and are the same regardless of which points A and B are used in the above equation. That is, a rigid body has one (and only one) angular velocity, and only one angular acceleration.

- The *direction* $\hat{k}$ for the angular velocity and acceleration vectors denotes the axis about which the body rotates. The signs of ω and α provide the *sense* of the rotational velocity and acceleration. Suppose that $\hat{k}$ points out of the page. Then, by the right-hand rule, a positive sign denotes counterclockwise motion, and a negative sign denotes clockwise rotation.

- If A and B both lie in a plane perpendicular to the $\hat{k}$ direction (i.e., the xy plane) then the last term on the right hand side of the acceleration equation simplifies to:

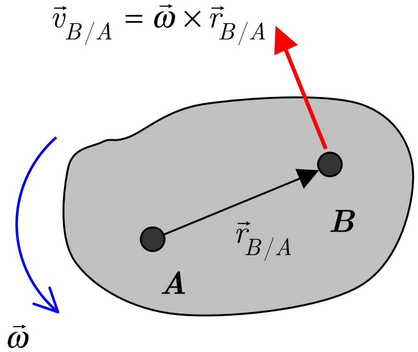
$$\begin{aligned}
\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) &= \left(\omega\hat{k}\right) \times \left[\left(\omega\hat{k}\right) \times \left(x\hat{\imath} + y\hat{\jmath}\right)\right] \\
&= \left(\omega\hat{k}\right) \times \left(x\omega\hat{\jmath} - y\omega\hat{\imath}\right) \\
&= -x\omega^2\hat{\imath} - y\omega^2\hat{\jmath} \\
&= -\omega^2\left(x\hat{\imath} + y\hat{\imath}\right) \\
&= -\omega^2\vec{r}_{B/A}
\end{aligned}$$

  In this case, the form of the acceleration simplifies to: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2\vec{r}_{B/A}$.

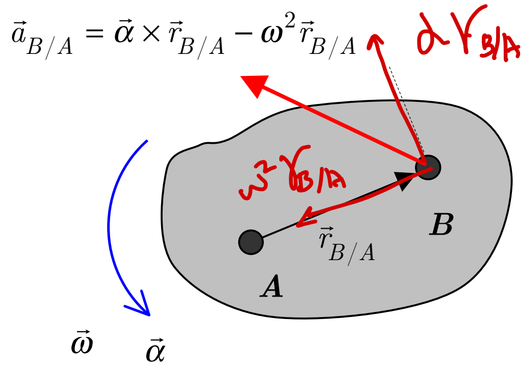
- If A is a fixed point ($\vec{v}_A = \vec{0}$), then from above we see that $\vec{v}_B = \vec{\omega} \times \vec{r}_{B/A}$. In this case, the body is rotating about point A. Since $\vec{\omega} \times \vec{r}_{B/A}$ is perpendicular to $\vec{r}_{B/A}$, $\vec{v}_B$ is perpendicular to $\vec{r}_{B/A}$ when the body is rotating about point A.

- For general motion of a body with no prescribed fixed points, the point about which the body rotates is not immediately obvious. We will address this point later on when we work with *instant centers.*

- **Interpretation**: The relative velocity vector $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \vec{\omega} \times \vec{r}_{B/A}$ is always perpendicular to the line connecting points A and B. However, the relative acceleration vector $\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = \vec{\alpha} \times \vec{r}_{B/A} - \omega^2\vec{r}_{B/A}$ is NOT perpendicular to the line connecting points A and B. How do we know that this is true? See the following figure.



$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$
$\vec{r}_{B/A}$
B
A
$\vec{\omega}$



$\vec{a}_{B/A} = \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$
$\alpha r_{B/A}$
$\omega^2 r_{B/A}$
$\vec{r}_{B/A}$
B
A
$\vec{\omega}$
$\vec{\alpha}$

Example 2.A.1

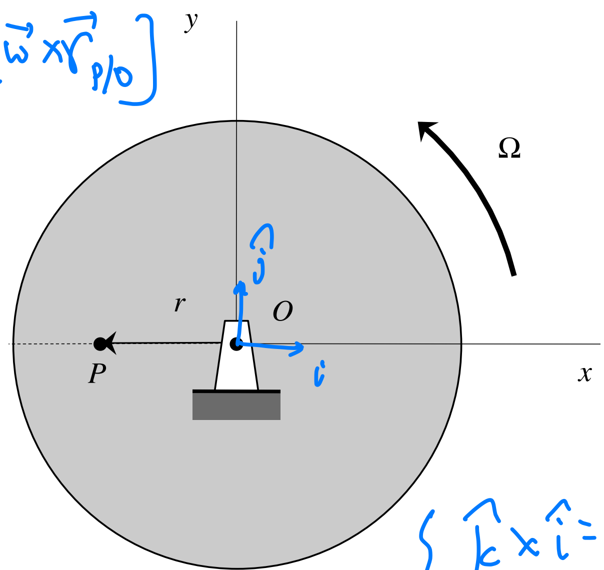
Given: The disk shown is rotating at a non-constant rate of Ω about a fixed axis passing through its center O. At a particular instant, the acceleration vector of point P on the disk is $\vec{a}_P$.

Find: Determine:

(a) The angular velocity of the disk at this instant; and
(b) The angular acceleration of the disk at this instant.

Use the following parameters in your analysis: $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2 and $r = 0.4$ m. Also, be sure to write your answers as vectors.

$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times \vec{r}_{p/o} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{p/o}]$$



$$\left\{ \begin{array}{l} \vec{a}_o = 0 \\ \vec{r}_{p/o} = -r\hat{i} \\ \vec{\alpha} = \alpha \cdot \hat{k} \\ \vec{\omega} = \omega \cdot \hat{k} \end{array} \right.$$

$$\left\{ \begin{array}{l} \hat{k} \times \hat{i} = \hat{j} \\ \hat{k} \times \hat{j} = -\hat{i} \end{array} \right.$$

$$\Rightarrow \vec{a}_p = 3\hat{i} + 4\hat{j} \ m/s^2$$

$$= \alpha[\hat{k} \times (-r\hat{i})] + \omega\hat{k} \times [\omega\hat{k} \times (-r\hat{i})]$$

$$= -\alpha r(\hat{k} \times \hat{i}) - \omega^2 r \, \hat{k} \times (\hat{k} \times \hat{i})$$

$$= \underbrace{-\alpha r}_{=4}\hat{j} + \underbrace{\omega^2 r}_{=3}\hat{i}$$

$$\Rightarrow \alpha = -\frac{4}{r}$$

$$\omega = \pm\sqrt{\frac{3}{r}}$$

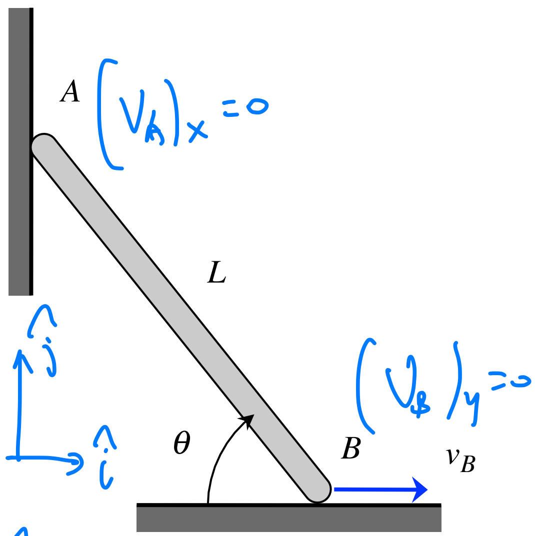
Example 2.A.7

Given: End B of the link moves to the right with a constant speed v_B. $\vec{a}_B = 0$

Find: Determine:
(a) The angular velocity of link AB; and
(b) The angular acceleration of link AB.

Use the following parameters in your analysis: $v_B = 3$ m/s, $L = 0.5$ m and $\theta = 36.87°$. Also, be sure to express your answers as vectors.

$\vec{V}_A = 0\,\hat{i} + V_A\,\hat{j}$ $\quad \vec{V}_B = V_B\,\hat{i} + 0\,\hat{j}$



$\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{A/B}$

$$\begin{cases} \vec{\omega} = \omega\hat{k}, \\ \vec{r}_{A/B} = -L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j} \end{cases}$$

$\Rightarrow \vec{V}_A = V_B\,\hat{i} + \omega\hat{k} \times (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$

$= V_B\,\hat{i} - \omega L\cos\theta\,\hat{j} - \omega L\sin\theta\,\hat{i}$

$= \underbrace{(V_B - \omega L\sin\theta)}_{=0}\hat{i} \; \underbrace{-\,\omega L\cos\theta}_{=V_A}\hat{j} \Rightarrow \begin{cases} \omega = \dfrac{V_B}{L\sin\theta} \\ V_A = -V_B\cot\theta \end{cases}$

$\vec{a}_A = \underbrace{\vec{a}_B}_{0} + \vec{\alpha} \times \vec{r}_{A/B} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})$

$= \alpha\hat{k} \times (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}) + \omega\hat{k} \times [\omega\hat{k} \times (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})]$

$= -\alpha L\cos\theta\,\hat{j} - \alpha L\sin\theta\,\hat{i} + \omega^2 L\cos\theta\,\hat{i} - \omega^2 L\sin\theta\,\hat{j}$

$= \underbrace{(\omega^2 L\cos\theta - \alpha L\sin\theta)}_{=0}\hat{i} + (-\omega^2 L\sin\theta - \alpha L\cos\theta)\,\hat{j}$

$\Rightarrow \alpha = \dfrac{\omega^2 \cos\theta}{\sin\theta} = \dfrac{V_B^2\cos\theta}{L^2\sin^3\theta}$