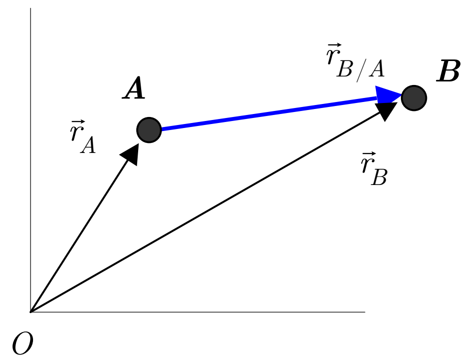
D. Kinematics: Relative and Constrained Motion

Background

By definition, the vector $\vec{r}_{B/A}$ is the vector which points FROM point A TO point B. From the vector diagram to the right, we see that $\vec{r}_{B/A}$ is related to the position vectors of points A and B as:

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

From this we see that $\vec{r}_{B/A}$ is the position of point B relative to the position of point A. Differentiation of this vector will produce the relative velocity and relative acceleration vectors of B with respect to A.



Objectives

The goals of this lecture are to:

- develop and use the relative motion kinematic equations relating the velocity and acceleration of two points
- study the relative kinematics of two points whose motion are constrained by connections of taut *inextensible* cables

Lecture Material

Relative Motion

As discussed above, the vector:

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

represents the position of point B relative to the position of point A.

If we take time derivatives of the above equation, we arrive at:

$$\frac{d}{dt}\vec{r}_{B/A} = \frac{d}{dt}\vec{r}_B - \frac{d}{dt}\vec{r}_A \qquad \Rightarrow \qquad \vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

and

$$\frac{d^2}{dt^2}\vec{r}_{B/A} = \frac{d^2}{dt^2}\vec{r}_B - \frac{d^2}{dt^2}\vec{r}_A \qquad \Rightarrow \qquad \vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

or relative to

where $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$ are interpreted as the *velocity of B with respect to A* and the *acceleration of B with respect to A*, respectively.

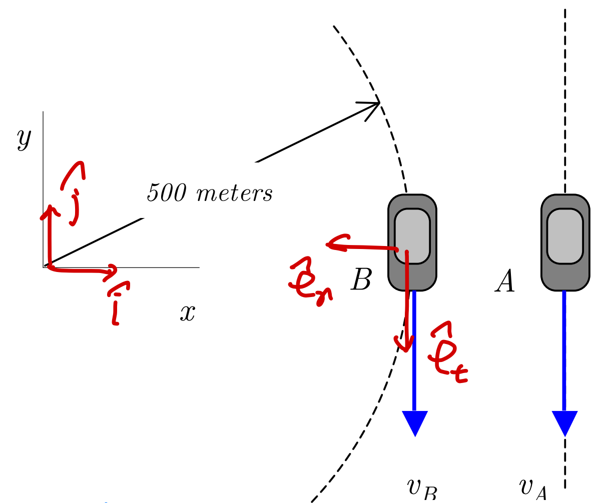
Chapter 1

Example 1.D.1 path is a circle of $\rho = 500$ m

Given: At the instant shown, car B is traveling with a speed of 50 km/hr and is slowing down at a rate of 10 km/hr^2. Car A is moving with a speed of 80 km/hr, a speed that is increasing at a rate of 10 km/hr^2. At this instant, A and B are traveling in the same direction.

Find: What acceleration does a passenger in car A observe for car B? $\vec{a}_{B/A}$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$



path geometry is known for B

→ Use path coordinates for $\vec{a}_B$

$$\vec{a}_B = \dot{v}_B \hat{e}_t + \frac{v_B^2}{\rho} \hat{e}_n$$

$\dot{v}_B = -10$ km/hr^2 , $v_B = 50$ km/hr , $\rho = 500$ m

$\vec{a}_A = -\dot{v}_A \hat{j}$, $\dot{v}_A = +10$ km/hr^2

→ Convert $\vec{a}_B$ into Cartesian coordinates,

at this instant: $\hat{e}_t = -\hat{j}$, $\hat{e}_n = -\hat{i}$

$$\Rightarrow \quad \vec{a}_B = -\frac{v_B^2}{\rho} \hat{i} - \dot{v}_B \hat{j}$$

$$\Rightarrow \quad \vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = -\frac{v_B^2}{\rho} \hat{i} + (\dot{v}_A - \dot{v}_B) \hat{j}$$

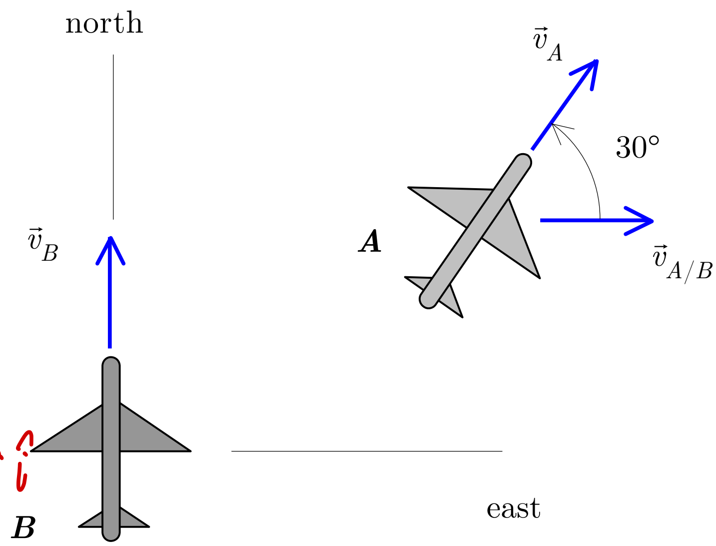
Example 1.D.2

$\vec{v}_B = v_B \hat{j}$

Given: Jet B is traveling due north with a speed of $v_B = 600$ km/hr. Passengers on jet B observe A to be flying sideways and moving due east.

Find: Determine:
(a) The speed of A; and
(b) The speed of A as observed by the passengers on jet B.

$\vec{v}_{A/B} = v_{A/B}\,\hat{i}$



$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B}$

$= v_{A/B}\cdot\hat{i} + v_B\,\hat{j}$

$= |\vec{v}_A|\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$\Rightarrow \dfrac{v_B}{|\vec{v}_A|} = \sin 30^\circ \qquad \Rightarrow |\vec{v}_A| = \dfrac{v_B}{\sin 30^\circ}$

$\Rightarrow v_{A/B} = |\vec{v}_A| \cos\theta$

What happens if $\theta = 90^\circ$?

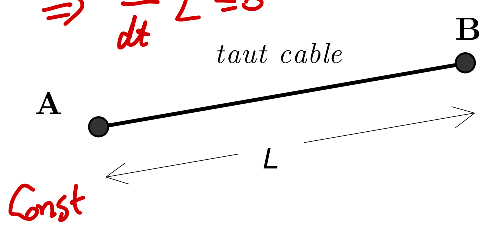
$\Rightarrow v_{A/B} = 0$

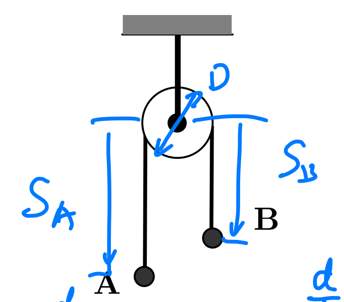
Constrained Motion

The above expressions are valid for relating the general motion of two points A and B. Often times the relative motion of these two points is constrained in such a way that the motion of one point is dependent on the motion of the second point. One example of this is when a taut, inextensible cable connects the two points.

use equations to describe the constraints

$L = \text{const} \Rightarrow \frac{d}{dt} L = 0$





$L = S_A + S_B + \frac{1}{2}\pi D = \text{Const} \Rightarrow S_A + S_B = \text{Const} \xrightarrow{\frac{d}{dt}} V_A + V_B = 0 \xrightarrow{\frac{d}{dt}} a_A + a_B = 0$

→ D doesn't affect the constraint eq. → $\boxed{\text{We can assume } D = 0}$

- Since the cable is inextensible, the length, L, of the cable does not change as the positions of points A and B change.
- Since a cable is flexible, it can be pulled over a pulley, as shown above right. Since the cable is inextensible, L remains constant for all motion of A and B so long as the cable remains taut. The distance between ends A and B does not, however, remain constant.

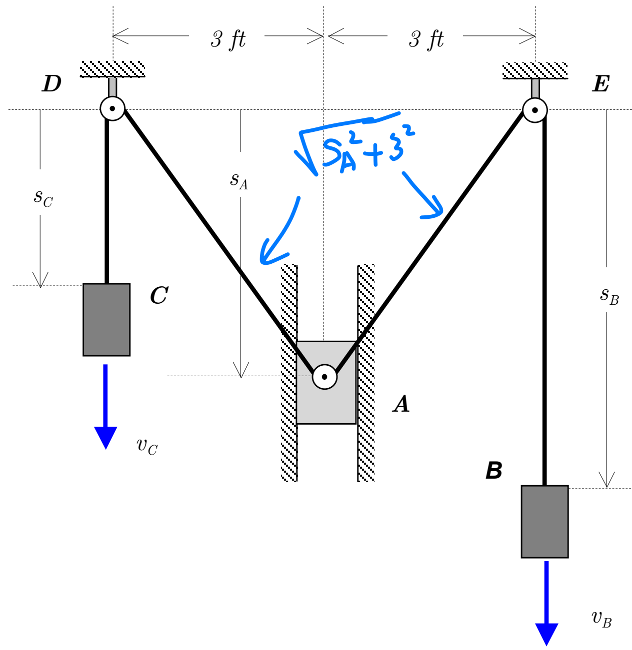
Based on these remarks, we can set up some general steps for solving problems for which particles are connected by taut, inextensible cables.

1. Carefully define a set of coordinates that describe the motion of the various particles in the system.
2. For each cable, write an expression for its length, L, in terms of an appropriate set of coordinates defined above in step 1.
3. Differentiate (with respect to time) the above expression for the cable length L and set $dL/dt = 0$ to determine the velocity constraint.
4. Differentiate again with respect to time to determine the acceleration constraint.
5. Repeat steps 2 through 4 for each cable in the system.

Example 1.D.6

Given: Blocks B and C are connected by a single inextensible cable, with this cable being wrapped around pulleys at D and E. In addition, the cable is wrapped around a pulley attached to block A as shown. Assume the radii of the pulleys to be small. Blocks B and C move downward with speeds of $v_B = 6$ ft/s and $v_C = 18$ ft/s, respectively.

Find: Determine the velocity of block A when $s_A = 4$ ft.



Cable length:

$$L = s_B + 2\sqrt{s_A^2+3^2} + s_C = \text{Const}$$

$\xrightarrow{\frac{d}{dt}}$

$$v_B + 2\frac{d}{dt}\left(\sqrt{s_A^2+3^2}\right) + v_C = 0$$

$$v_B + 2\frac{s_A}{\sqrt{s_A^2+3^2}} \cdot v_A + v_C = 0$$

$$\Rightarrow v_A = -(v_B + v_C) \cdot \frac{\sqrt{s_A^2+3^2}}{2 s_A}$$