C. Joint Description – Combined Usage of the Cartesian, Path and Polar Descriptions

As previously noted, we have the following three descriptions for the kinematics for the motion of points moving in the plane:

velocity vector

$$\begin{aligned} \vec{v} &= \dot{x}\,\hat{i} + \dot{y}\,\hat{j} && ;\ \textit{Cartesian} \\ &= v\,\hat{e}_t && ;\ \textit{path} \\ &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta && ;\ \textit{polar} \end{aligned}$$

acceleration vector

$$\begin{aligned} \vec{a} &= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} && ;\ \textit{Cartesian} \\ &= \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n && ;\ \textit{path} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta && ;\ \textit{polar} \end{aligned}$$

FACT: Regardless of which description used, VELOCITY = VELOCITY (and, ACCELERATION = ACCELERATION). That is, all three descriptions for velocity (and acceleration) are referring to the same velocity (and acceleration) vector, and all descriptions are equivalent. The descriptions just look different and contain different information on the motion.

In many problems, we will need to work with kinematic descriptions using two or more distinct coordinate systems. For these problems, we will inevitably need to convert among these different kinematic descriptions.

There are two critical steps in solving for components for one kinematic description in terms of another: (i) writing unit vectors of one description in terms of the other description, and (ii) using either projection methods or coefficient balancing for like kinematic descriptions. Consider the following motivating examples.

Example:

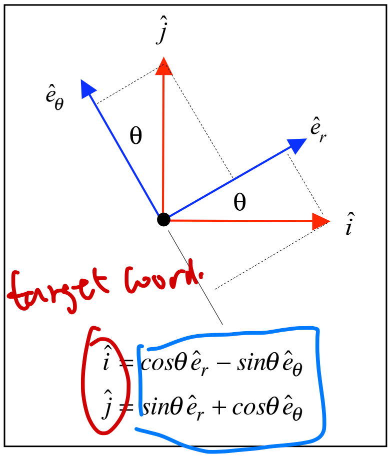
Motion of a car is described in $x(t)\hat{i} + y(t)\hat{j}$

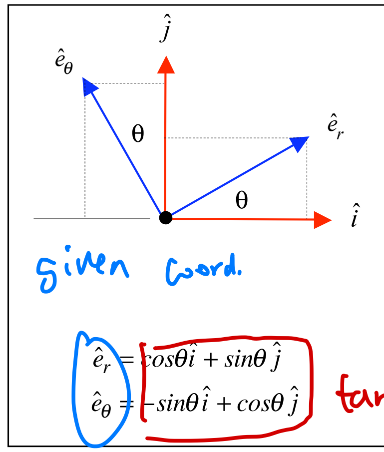
→ what is · rate of speed change

· side-way acceleration

— path coordinates

MOTIVATING EXAMPLE:
Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$.





target coord.

given coord.

① Method of projection

From the figure on the left:

$$\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta = 0.8\hat{e}_r - 0.6\hat{e}_\theta$$

$$\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta = 0.6\hat{e}_r + 0.8\hat{e}_\theta$$

$\hat{e}_t \cdot \hat{e}_t = 1$, $\hat{e}_t \cdot \hat{e}_n = 0$

Therefore,

$$\dot{x} = \hat{i} \bullet \vec{v} = (0.8\hat{e}_r - 0.6\hat{e}_\theta) \bullet (10\hat{e}_r - 20\hat{e}_\theta) = (0.8)(10) + (-0.6)(-20) = 20 \text{ m/s}$$

$$\dot{y} = \hat{j} \bullet \vec{v} = (0.6\hat{e}_r + 0.8\hat{e}_\theta) \bullet (10\hat{e}_r - 20\hat{e}_\theta) = (0.6)(10) + (0.8)(-20) = -10 \text{ m/s}$$

$$\ddot{x} = \hat{i} \bullet \vec{a} = (0.8\hat{e}_r - 0.6\hat{e}_\theta) \bullet (3\hat{e}_r + 2\hat{e}_\theta) = (0.8)(3) + (-0.6)(2) = 1.2 \text{ m/s}^2$$

$$\ddot{y} = \hat{j} \bullet \vec{a} = (0.6\hat{e}_r + 0.8\hat{e}_\theta) \bullet (3\hat{e}_r + 2\hat{e}_\theta) = (0.6)(3) + (0.8)(2) = 3.4 \text{ m/s}^2$$

② Method of substitution :

An alternate approach is to write the following (using the figure on the right):

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j} = 0.8\hat{i} + 0.6\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j} = -0.6\hat{i} + 0.8\hat{j}$$

Therefore,

$$\vec{v} = 10\hat{e}_r - 20\hat{e}_\theta = 10(0.8\hat{i} + 0.6\hat{j}) - 20(-0.6\hat{i} + 0.8\hat{j}) = (20\hat{i} - 10\hat{j}) \text{ m/s}$$

$$\vec{a} = 3\hat{e}_r + 2\hat{e}_\theta = 3(0.8\hat{i} + 0.6\hat{j}) + 2(-0.6\hat{i} + 0.8\hat{j}) = (1.2\hat{i} + 3.4\hat{j}) \text{ m/s}^2$$

Balancing coefficients gives: $\dot{x} = 20$ m/s, $\dot{y} = -10$ m/s, $\ddot{x} = 1.2$ m/s^2 and $\ddot{y} = 3.4$ m/s^2. These results agree with those found from the projection approach highlighted above.

MOTIVATING EXAMPLE:
Suppose the velocity and acceleration of a particle are known in terms of their Cartesian components as: $\vec{v} = (30\hat{\imath} - 40\hat{\jmath})$ m/s and $\vec{a} = (-10\hat{\jmath})$ m/s^2. From this, we want to find the speed v, rate of change of speed $\dot{v}$ and the radius of curvature ρ of the path of the particle (path description variables).

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{30^2 + 40^2} = 50 \text{ m/s}$$

Since $\vec{v} = v\hat{e}_t$, we can write:

$$\boxed{\hat{e}_t = \frac{\vec{v}}{v} = \frac{30\hat{\imath} - 40\hat{\jmath}}{50} = 0.6\hat{\imath} - 0.8\hat{\jmath}}$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = -10\hat{\jmath}$$

$$\dot{v} = \hat{e}_t \cdot \vec{a}$$

From this, we can write:

$$\dot{v} = \hat{e}_t \bullet \vec{a} = (0.6\hat{\imath} - 0.8\hat{\jmath}) \bullet (-10\hat{\jmath}) = 8 \text{ m/s}^2$$

Since $\vec{a} = \dot{v}\hat{e}_t + (v^2/\rho)\hat{e}_n$, we can write:

$$|\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \quad \Rightarrow \quad \rho = \frac{v^2}{\sqrt{|\vec{a}|^2 - \dot{v}^2}} = \frac{50^2}{\sqrt{10^2 - 8^2}} = \frac{2500}{6} \text{ m}$$

or $\frac{v^2}{\rho} = \hat{e}_n \cdot \vec{a} \Rightarrow \rho$

The motivating examples above demonstrate the thought processes for both the projection and coefficient balancing methods. Use these examples as a guide for solving the joint kinematics problems that you encounter. Consider, in particular, the second example where the method of determining the path tangent unit vector directly from the velocity vector: $\hat{e}_t = \vec{v}/v$. You will find this result to be useful in many problems involving path coordinates. Also, consider the following projection results for determining Cartesian and polar components of velocity and acceleration:

$$\dot{x} = \vec{v} \bullet \hat{\imath} \qquad \dot{y} = \vec{v} \bullet \hat{\jmath}$$
$$\ddot{x} = \vec{a} \bullet \hat{\imath} \qquad \ddot{y} = \vec{a} \bullet \hat{\jmath}$$

where $\vec{v}$, $\vec{a}$, $\hat{\imath}$ and $\hat{\jmath}$ are written in terms of either polar or path components, and

$$\dot{r} = \vec{v} \bullet \hat{e}_r \qquad r\dot{\theta} = \vec{v} \bullet \hat{e}_\theta$$
$$\ddot{r} - r\dot{\theta}^2 = \vec{a} \bullet \hat{e}_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \vec{a} \bullet \hat{e}_\theta$$

where $\vec{v}$, $\vec{a}$, $\hat{e}_r$ and $\hat{e}_\theta$ are written in terms of either Cartesian or path components.

Do NOT attempt to memorize these results. Work to understand the kinematic fundamentals and the processes behind the methods.

Example 1.C.2

Given: Pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant shown, guide A moves to the right with a speed of v_A, a speed that is changing at a rate of $\dot{v}_A$. At the same time, B is moving downward with a speed of v_B with a rate of change of speed of $\dot{v}_B$.

Find:

$\dot{v}$

(a) The rate of change of speed of P at this instant; and

(b) The radius of curvature ρ of the path followed by P at this instant.

Use the following parameters: $v_A = 0.2$ m/s, $v_B = 0.15$ m/s, $\dot{v}_A = 0.75$ m/s^2 and $\dot{v}_B = 0$.

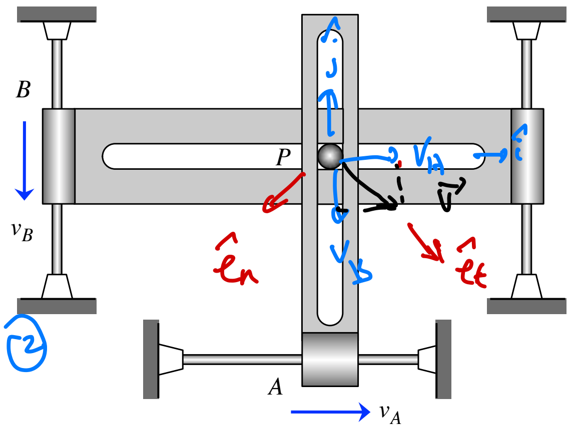
We need path coordinates:

$$\vec{v} = v\hat{e}_t = \dot{x}\hat{i} + \dot{y}\hat{j} \quad (1)$$

$$\dot{x} = v_A \quad , \quad \dot{y} = -v_B$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = \ddot{x}\hat{i} + \ddot{y}\hat{j} \quad (2)$$

$$\ddot{x} = \dot{v}_A \quad , \quad \ddot{y} = -\dot{v}_B = 0$$



$$(1) \Rightarrow v = \sqrt{v_A^2 + (-v_B)^2} \quad , \quad \hat{e}_t = \frac{v_A\hat{i} - v_B\hat{j}}{\sqrt{v_A^2 + v_B^2}}$$

$$(2) \Rightarrow \dot{v} = \vec{a}\cdot\hat{e}_t = (\dot{v}_A\hat{i} - \dot{v}_B\hat{j})\cdot\left(\frac{v_A\hat{i} - v_B\hat{j}}{\sqrt{v_A^2+v_B^2}}\right)$$

$$= \frac{1}{\sqrt{v_A^2+v_B^2}}\left(v_A\dot{v}_A + v_B\underbrace{\dot{v}_B}_{0}\right)$$

$$(2) \Rightarrow |\vec{a}|^2 = \underbrace{\dot{v}^2 + \frac{v^4}{\rho^2}}_{\text{from path coord.}} = \underbrace{\dot{v}_A^2 + (-\dot{v}_B)^2}_{\text{from Cartesian.}} \Rightarrow \rho = \sqrt{\frac{v^4}{\dot{v}_A^2 + \dot{v}_B^2 - \dot{v}^2}}$$

Example 1.C.4

$\dot{v}_p = 0$

Given: At the bottom of a loop, an airplane P has a constant speed of v_P with the radius of curvature for the aircraft being ρ. The airplane is at a radial distance of r and at an angle of θ from a radar tracking station at O.

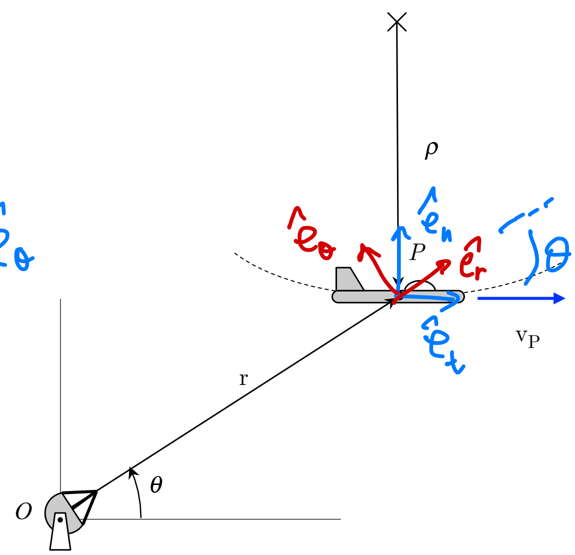
Find: Determine numerical values for $\ddot{r}$ and $\ddot{\theta}$ at this instant in time.

Use the following: $v_P = 75$ m/s, $\rho = 3000$ m, $r = 1000$ m and $\theta = 36.87°$.

$$\vec{v} = v\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \underbrace{\dot{v}_p}_{=0}\hat{e}_t + \frac{v_p^2}{\rho}\hat{e}_n \quad \text{known}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$



From geometry:

$$\begin{cases} \hat{e}_r = \cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n \\ \hat{e}_\theta = -\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n \end{cases}$$

Velocity: $\rightarrow$

$$\begin{cases} \dot{r} = \vec{v}\cdot\hat{e}_r = v\hat{e}_t)\cdot(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n) = v\cos\theta \\ r\dot{\theta} = \vec{v}\cdot\hat{e}_\theta = (v\hat{e}_t)\cdot(-\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n) = -v\sin\theta \end{cases}$$

$\longrightarrow$ Calculate $\dot{r}$, $\dot{\theta}$

acceleration :

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})(-\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n)$$

rearrange $\longrightarrow$

$$= \boxed{\left[(\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\right]}\hat{e}_t = \dot{v}_p$$

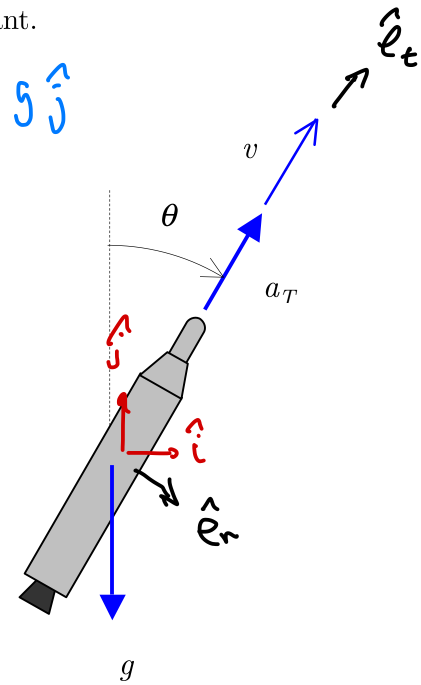
$$+ \boxed{\left[(\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta\right]}\hat{e}_n = \frac{v_p^2}{\rho}$$

$\longrightarrow$ 2 Equations , 2 unknowns

Example 1.C.1

Given: A rocket is traveling at an altitude at which the gravitational acceleration is known to be $g = 26.5$ ft/s^2. The thrust force on the rocket produces an acceleration of $a_T = 29.3$ ft/s^2 along the axis of the rocket. At the position shown ($\theta = 36.87°$) the speed of the rocket is known to be $v = 2800$ ft/s.

Find: Determine:
(a) The rate of change of speed of the rocket at this instant; and
(b) The radius of curvature for the rocket's path at this instant.



Total acceleration: $\vec{a} = \vec{a}_T + \vec{g} = a_T \hat{e}_t - g\hat{j}$

in path Coordinates: $= \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

express $\hat{j}$ in $\hat{e}_t$, $\hat{e}_n$:

$$\hat{j} = \cos\theta\,\hat{e}_t - \sin\theta\,\hat{e}_n$$

$$\Rightarrow \vec{a} = a_T\hat{e}_t - g\cos\theta\,\hat{e}_t + g\sin\theta\,\hat{e}_n$$

$$= \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$\begin{cases} \hat{e}_t: & a_T - g\cos\theta = \dot{v} \\ \hat{e}_n: & g\sin\theta = \dfrac{v^2}{\rho} \end{cases}$$

$$\Rightarrow \quad \dot{v} = a_T - g\cos\theta$$

$$\rho = \frac{v^2}{g\sin\theta}$$