Polar Kinematics

For the polar description, the following set of unit vectors will be used:

functions of t

- $\hat{e}_r$: pointing from O to point P
- $\hat{e}_\theta$: perpendicular to $\hat{e}_r$ and pointing in the "positive θ direction" (see the figure below)

Here we write the position vector of P as:

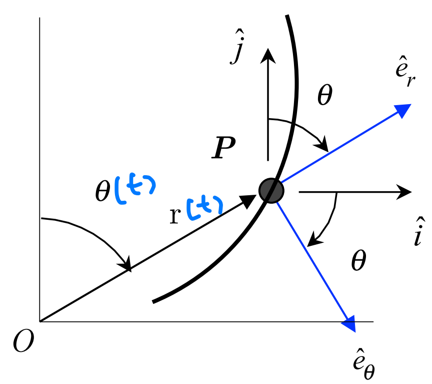
$$\vec{r} = r\hat{e}_r \qquad \vec{r}(t) = r(t)\cdot\hat{e}_r(t)$$

The velocity of point P is given by the first time derivative of the position vector for P:

$$\vec{v} = \frac{d}{dt}\left(r\hat{e}_r\right) = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} \qquad \text{(product rule of differentiation)}$$

$$= \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} \qquad \text{(chain rule of differentiation)}$$

$= \dot{\theta}$ → rate of angle change or angular speed



$$\hat{e}_r = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$\hat{e}_\theta = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} \qquad \Rightarrow$$

$$\boxed{\begin{aligned}\frac{d\hat{e}_r}{d\theta} &= \cos\theta\,\hat{i} - \sin\theta\,\hat{j} = \hat{e}_\theta \\ \frac{d\hat{e}_\theta}{d\theta} &= -\sin\theta\,\hat{i} - \cos\theta\,\hat{j} = -\hat{e}_r\end{aligned}}$$

Using the equations in the figure above, we can now write the velocity vector for P as:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Using the product rule of differentiation on the velocity vector above, we obtain:

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\hat{e}_\theta\right) = \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}\left(\hat{e}_r\right) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}\left(\hat{e}_\theta\right)$$

From above, we know that $\frac{d}{dt}\left(\hat{e}_r\right) = \dot{\theta}\hat{e}_\theta$, and through the use of the chain rule and the figure above, it can be shown that:

$$\frac{d}{dt}\left(\hat{e}_\theta\right) = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r$$

Therefore, we have:

$$
\begin{aligned}
\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}\left(\dot{\theta}\hat{e}_\theta\right) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\left(-\dot{\theta}\hat{e}_r\right) \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta
\end{aligned}
$$

— does not have intuitive physical interpretations implicit: $r = r(\theta)$

$\Rightarrow \dot{r} = \frac{dr}{d\theta} \cdot \dot{\theta}$

$\Rightarrow \ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\theta}\dot{\theta}\right) = \frac{d^2 r}{d\theta^2} \cdot \dot{\theta}^2 + \frac{dr}{d\theta} \cdot \ddot{\theta}$

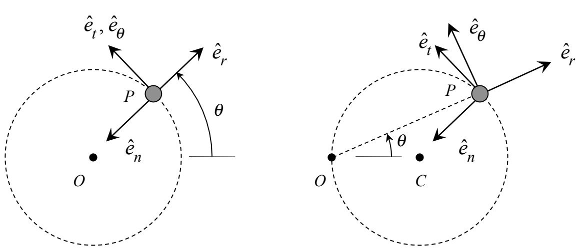
Discussion – Polar Description

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

- *The values for the components of these vectors depend on your choice of the point O.* Therefore, you need to carefully define your choice of point O at the beginning of the problem and stick with it throughout the problem.
- When the path of P is given as $r = r(\theta)$ you will need to use the *chain rule of differentiation* to find the time derivatives $\dot{r} = dr/dt$ and $\ddot{r} = d^2r/dt^2$ in terms of the time derivatives $\dot{\theta}$ and $\ddot{\theta}$.
- These vector expressions use the components of velocity and acceleration projected on the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$. These projections are usually more difficult to determine than, say, the Cartesian projections and have less physical significance than the path component projections. However, in many applications involving observers of motion, the polar expressions are very useful.

CHALLENGE QUESTION: The path unit vectors ($\hat{e}_t$ and $\hat{e}_n$) share characteristics with the polar unit vectors ($\hat{e}_r$ and $\hat{e}_\theta$) in that they move along with the particle and they change orientation as the particle moves along its path. Can the two sets of unit vectors ever be aligned with each other?

ANSWER: As we know, the path unit vectors $\hat{e}_t$ and $\hat{e}_n$ are defined by the path. On the other hand, the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ depend on your choice of the observer O and the position of the particle relative to O. One special case when they are somewhat aligned is when the particle travels on a circular path with the observer O at the center of the circle, as shown in the figure below left. Here, $\hat{e}_t$ and $\hat{e}_\theta$ are aligned for all motion; $\hat{e}_r$ points outward from O and $\hat{e}_n$ points inward toward the center of the path O. To emphasize how the orientation of $\hat{e}_r$ and $\hat{e}_\theta$ depends on the choice of O, consider moving O to another location, as shown below right. Here the two sets of unit vectors are not aligned.

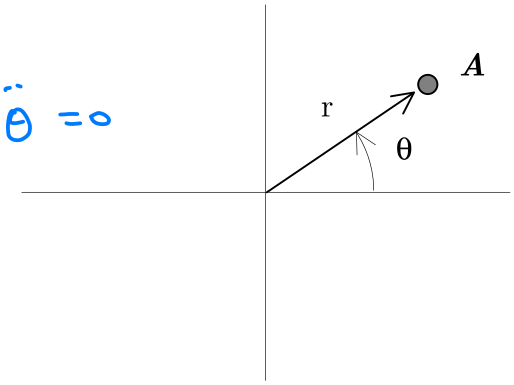


Example 1.A.4

Given: Particle A travels on a path such that the radial position of A is given by $r = 5\theta$, where r is given in meters and θ in radians. It is also known that $\dot{\theta} = 2$ rad/s $=$ *constant.*

Find: Determine:
(a) The velocity vector for A when $\theta = \pi$; and
(b) The acceleration vector for A when $\theta = \pi$.

given
$$\begin{cases} r(\theta) = 5\theta \\ \dot{\theta} = \text{const.} = 2 \text{ rad/s} \Rightarrow \ddot{\theta} = 0 \end{cases}$$



(a)
$$\vec{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\dot{r} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = 5\dot{\theta}$$

Substitute: $r = 5\theta$, $\theta = \pi$, $\dot{\theta} = 2$ rad/s

(b)
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (\underbrace{r\ddot{\theta}}_{0} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

from (a): $\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt}$

→ only need to find $\ddot{r}$

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right) = \underbrace{\frac{d^2r}{d\theta^2}}_{0}\dot{\theta}^2 + \frac{dr}{d\theta}\underbrace{\ddot{\theta}}_{0} = 0$$

Question C1.6

Particle P travels on a path described by the polar coordinates $r = 2\cos\theta$, where r is given in feet and θ is given in radians. When $\theta = \pi/3$ radians, it is known that $\dot{\theta} = -3$ rad/s and $\ddot{\theta} = 0$.

At this instant (circle the correct answer):

(a) $\dot{r} < 0$

(b) $\dot{r} = 0$

(c) $\dot{r} > 0$

$$\dot{r} = \frac{d}{dt}\left(r(\theta)\right) = \frac{dr}{d\theta}\cdot\dot{\theta} = -2\sin\theta\cdot\dot{\theta}$$

$$\sin\theta\Big|_{\theta=\frac{\pi}{3}} > 0, \quad \dot{\theta} = -3 < 0$$

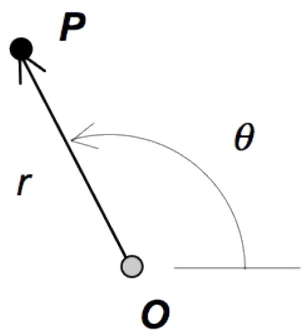
$$\Rightarrow \dot{r} > 0$$

At this instant (circle the correct answer):

(a) $\ddot{r} < 0$

(b) $\ddot{r} = 0$

(c) $\ddot{r} > 0$



$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}\left(-2\sin\theta\,\dot{\theta}\right)$$

$$= -2\frac{d}{dt}(\sin\theta)\cdot\dot{\theta} - 2\sin\theta\,\ddot{\theta}$$

$$= -2\underbrace{\cos\theta}_{>0}\cdot\underbrace{\dot{\theta}^2}_{>0} - \underbrace{2\sin\theta\,\ddot{\theta}}_{=0 \text{ Since } \ddot{\theta}=0}$$

$$\Rightarrow \ddot{r} < 0$$