Path Kinematics

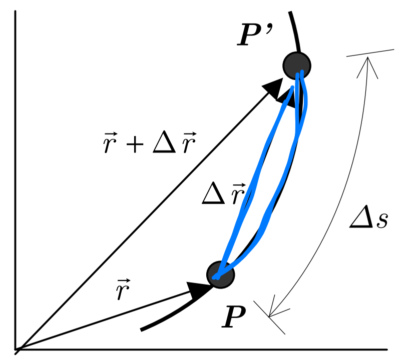
path geometry is known : $\vec{r} = \vec{r}(s)$

$s = s(t)$

Recall that for the path description, the position of particle P is given by a distance s measured along the path of P. The velocity of point P is given by the first time derivative of the position vector for P:

arc length

at $t+\Delta t$



at t

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\frac{d\vec{r}}{ds}$$

where the *chain rule of differentiation* has been used to introduce the path distance s into the kinematics and where $v = ds/dt$ is the "speed" of the particle.

scalar

magnitude of $\vec{v}$

By definition, the derivative $\frac{d\vec{r}}{ds}$ is:

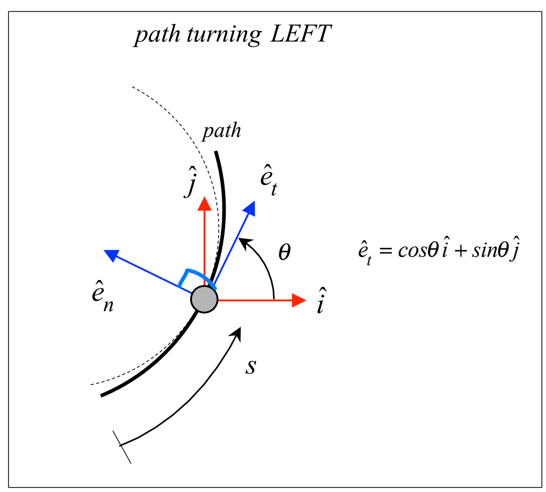
$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\vec{r}}{\Delta s}$$

where $\Delta\vec{r}$ is the change in the position vector $\vec{r}$ as the particle moves a distance s along its path (as shown in the figure where P has moved to P').

Two observations on the above derivative, $\frac{d\vec{r}}{ds}$:

- As $\Delta s \to 0$, the chord length $|\Delta\vec{r}|$ tends to the length of the arc length Δs (see above figure): $|\Delta\vec{r}| \to \Delta s$. Therefore, $d\vec{r}/ds$ is a "unit vector" (magnitude of "1").
- As $\Delta s \to 0$, the vector $\Delta\vec{r}$ becomes tangent to the path of P (see above figure).

direction of velocity



From this we conclude that $d\vec{r}/ds = \hat{e}_t$ = unit vector that is tangent to the path of P. Therefore,

$$\vec{v} = v\hat{e}_t$$

Basis vector:
- Cartesian: $\hat{i}, \hat{j}$ — not functions of t
- path: $\hat{e}_t, \hat{e}_n$ — both functions of t

Differentiation of the above with respect to time gives:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left(v\hat{e}_t\right)$$

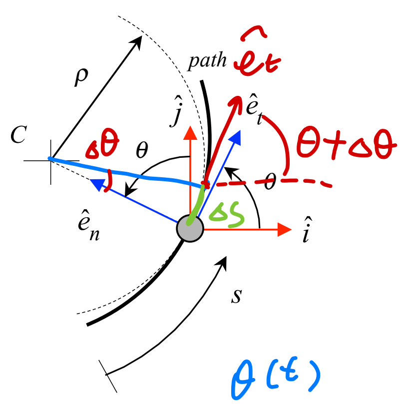
$$= \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \qquad \text{(product rule of differentiation)}$$

$$= \underbrace{\dot{v}}_{\dot{s}}\hat{e}_t + v\frac{d\hat{e}_t}{ds}\underbrace{\frac{ds}{dt}}_{=v} \qquad \text{(chain rule of differentiation)}$$

$$\frac{d\hat{e}_t}{ds} = \frac{d\hat{e}_t}{d\theta}\cdot\frac{d\theta}{ds}$$

$$= \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} \qquad \text{(chain rule of differentiation and } v = \frac{ds}{dt}\text{)}$$

path turning LEFT



$$\hat{e}_n = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

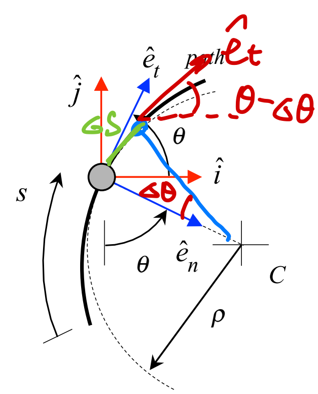
$$\hat{e}_t = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \Rightarrow$$

$$\frac{d\hat{e}_t}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_n$$

$$ds = +\rho\, d\theta \quad \Rightarrow \quad \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\therefore \frac{de_t}{d\theta}\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

path turning RIGHT



$$\hat{e}_n = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$$

$$\hat{e}_t = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \Rightarrow$$

$$\frac{d\hat{e}_t}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = -\hat{e}_n$$

$$ds = -\rho\, d\theta \quad \Rightarrow \quad \frac{d\theta}{ds} = -\frac{1}{\rho}$$

$$\therefore \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

C = "*center of curvature*" (*center of circle tangent to path having the same curvature as the path*)

ρ = "*radius of curvature*" *of path*

$$\frac{d\hat{e}_t}{ds} = \frac{\hat{e}_n}{\rho}$$

Consider the figures provided above showing the directions of the unit vectors $\hat{e}_t$ and $\hat{e}_t$ for left-turning and right-turning paths. From these figures, we see that for both cases:

$$\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{1}{\rho}\hat{e}_n$$

where ρ is the radius of curvature of the path. Using this relationship in the above gives:

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

which describes the acceleration of a particle in terms of its path components.

What does it mean?

Discussion – Path Description

We have seen that the velocity and acceleration of a particle can be written in terms of its path components by the following equations:

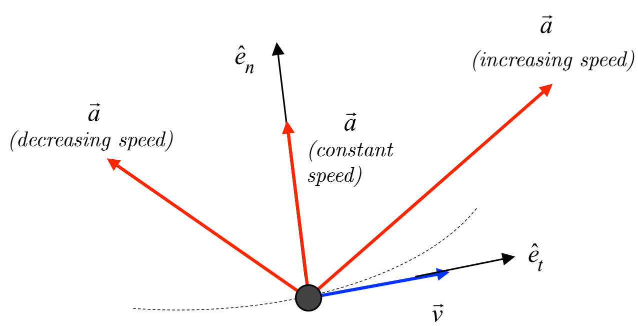
$$\vec{v} = v\hat{e}_t$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

- The velocity of a point is ALWAYS tangent to the path of the point. The magnitude of the velocity vector is the known as the scalar "speed" v of the point.

- The acceleration of the point has two components:
  - The component $\left(v^2/\rho\right)\hat{e}_n$ is *normal* to the path. This is commonly referred to as the "centripetal" component of acceleration. This component is ALWAYS directed inward to the path (*positive n-component*) since $v^2/\rho > 0$.

  turn at constant speed
  - The component $\dot{v}\hat{e}_t$ is tangent to the path. The magnitude of this component is the "rate of change of speed" $\dot{v}$ for the point.
    * When $\dot{v} > 0$ (increasing speed), the acceleration vector has a *positive t-component* (i.e., forward of $\hat{e}_n$).
    * When $\dot{v} = 0$ (constant speed), the acceleration vector has a *zero t-component* (i.e., $\vec{a}$ is aligned with $\hat{e}_n$). Note that constant speed does NOT imply zero acceleration!
    * When $\dot{v} < 0$ (decreasing speed), the acceleration vector has a *negative t-component* (i.e., backward of $\hat{e}_n$). See figure below.



- ρ is the radius of curvature for the path of the particle. If the path is known to be circular, ρ is the radius of the circle. For a general path known in terms of its Cartesian coordinates $y = y(x)$, the radius of curvature can be calculated from:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

Sharper turn → $\rho \searrow$

- The magnitude of the acceleration is given by the square root of the sum of the squares of its path components:

$$|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$$

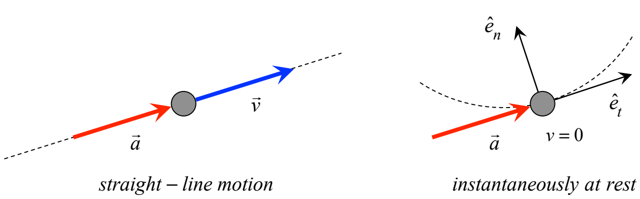
Do not confuse the terms "rate of change of speed" and "magnitude of acceleration":

  - Rate of change of speed $\dot{v}$ (as the name indicates) is the rate at which the speed changes in time; it is simply the tangential component of acceleration.

  - The magnitude of acceleration $|\vec{a}|$ accounts for both the tangential and normal components of acceleration, as shown in the above equation.

CHALLENGE QUESTION: The acceleration vector is the time derivative of the velocity vector: $\vec{a} = d\vec{v}/dt$. In contrast, the scalar rate of change of speed is the time derivative of the scalar speed: $\dot{v} = dv/dt$. As discussed above, the magnitude of acceleration is generally not the same as the magnitude of the rate of change of speed: $|\vec{a}| \neq |\dot{v}|$. Are there situations in which they are the same?

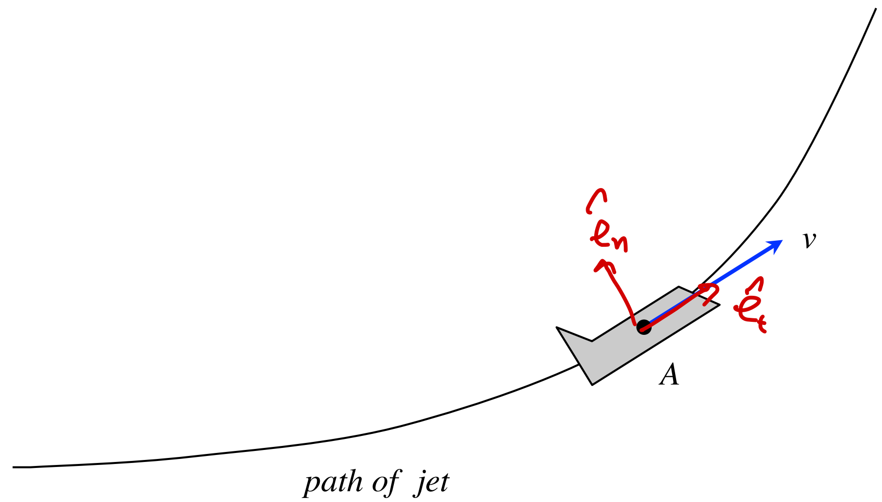
ANSWER: Since $|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$, $|\vec{a}| = |\dot{v}|$ ONLY IF $v^2/\rho = 0$. This occurs when either: (i) $\rho = \infty$ (straight-line, or rectilinear, motion), or (ii) $v = 0$ (particle instantaneously at rest). These two situations are shown in the following figure.



Example 1.A.3

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

Find: The radius of curvature of the path of the jet at A.



$$\vec{a} = \underbrace{\dot{v}}_{a_t = 5\,m/s^2}\hat{e}_t + \underbrace{\frac{v^2}{\rho}}_{a_n}\cdot\hat{e}_n$$

$$|\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2$$

$$\Rightarrow \frac{v^2}{\rho} = \sqrt{|\vec{a}|^2 - \dot{v}^2} \rightarrow \text{has to be positive}$$

$$\Rightarrow \rho = \frac{v^2}{\sqrt{|\vec{a}|^2 - \dot{v}^2}} \leftarrow$$

Substitude:

$v = 600$ km/hr, $\dot{v} = 5$ m/s^2,

$|\vec{a}| = 2.5g$