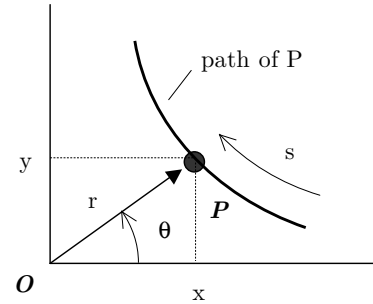


## A. Planar Kinematics: Cartesian, Path and Polar Coordinates

### Background

Point P moves on a curvilinear path in a plane. We will consider this planar motion of P using three different descriptions (see figure to the right):

- *Cartesian description* – here the path of P is known in terms of the Cartesian components  $x$  and  $y$ . Typically this path is given by an equation such as  $y = y(x)$  that relates the  $x$  and  $y$  components.
- *Path description* – here the position is known in terms of a distance  $s$  measured along the path of the particle.
- *Polar description* – here the position is known in terms of a radial distance  $r$  (as measured from point O) and the angle  $\theta$  for the line OP. The path of P is often expressed in terms of an equation such as  $r = r(\theta)$  that relates  $r$  and  $\theta$ .



### Objectives

The goal of this lecture is to write the velocity and acceleration for the planar motion of a point in terms of three alternate descriptions: Cartesian, path and polar. The results of these three kinematic descriptions will be compared and contrasted exposing the attributes of each.

## Lecture Material

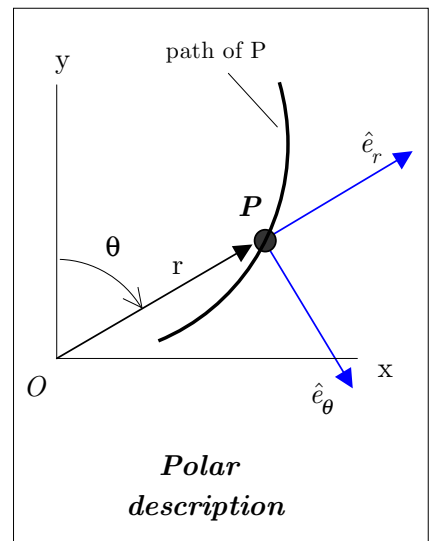
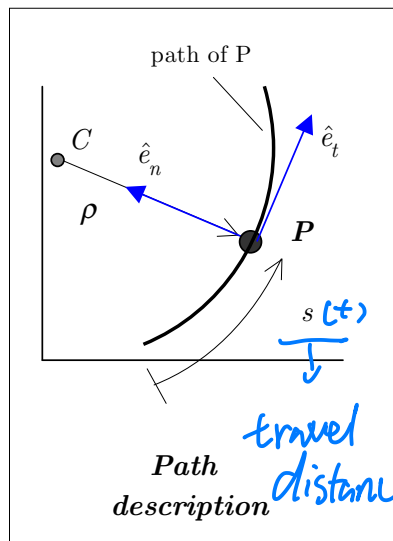
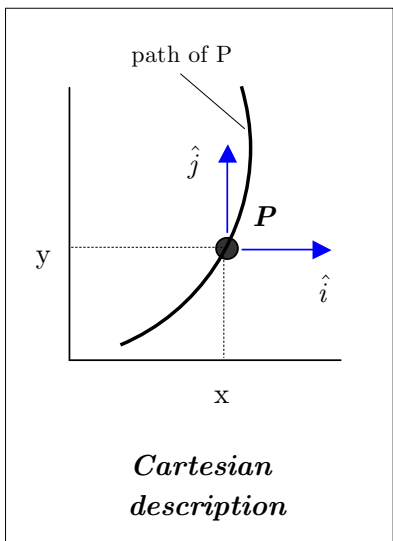
As always, the velocity and acceleration of point P are given by the first and second time derivatives, respectively, of the position vector  $\vec{r}$  for P:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

- Don't think  $\vec{r}$ ,  $\vec{v}$ ,  $\vec{a}$  as individual variables
- know one of them  $\xrightarrow{\frac{d}{dt}, \int}$  know all three

The kinematic equations for the Cartesian, path and polar descriptions are derived in the following notes. The following figures showing the kinematic variables and unit vectors will be used in these derivations.



- Intuitive

- motion:

$$\rightarrow \begin{cases} x(t) \\ y(t) \end{cases}$$

- Example:

Drive on a given route (path).

- path is known

- motion:

$\begin{cases} \text{path geometry} \\ \text{distance } s(t) \end{cases}$

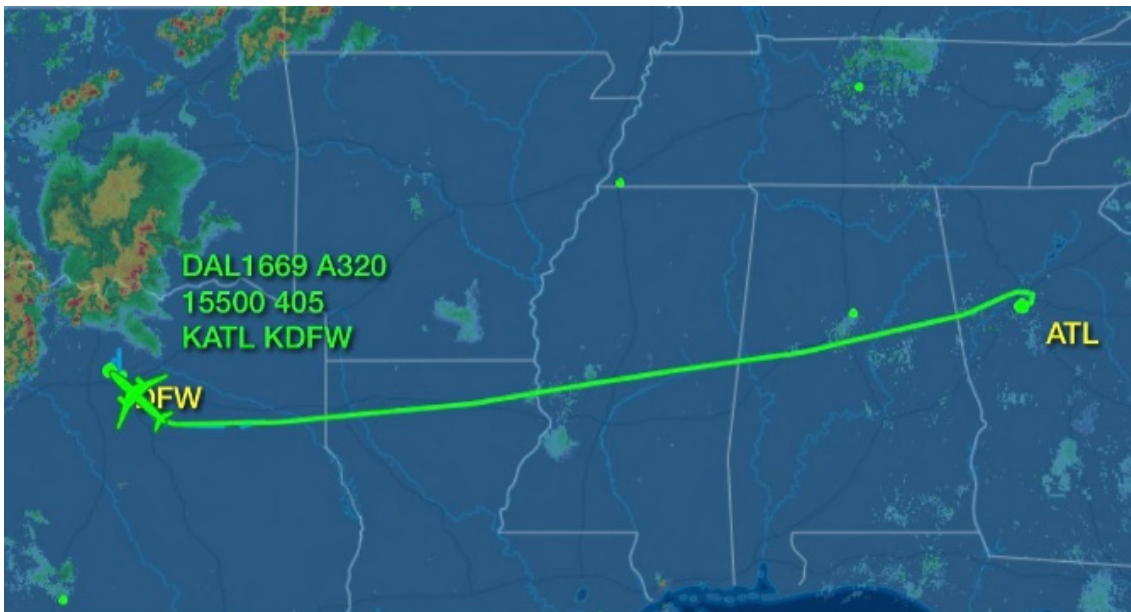
- Example:

Swing ride

- motion:

$$\begin{cases} \theta(t) \\ r(t) \end{cases}$$

## Flight tracking



## Swing ride



### Cartesian Kinematics

Here we write the position vector for P in terms of its  $x$  and  $y$  components and corresponding unit vectors:

$$\vec{r} = x\hat{i} + y\hat{j} \rightarrow \begin{cases} x(t), y(t) \\ \hat{i}, \hat{j} \end{cases} \text{ - not functions of } t$$

Let's assume that  $\hat{i}$  and  $\hat{j}$  represent constant directions (that is,  $\frac{d\hat{i}}{dt} = \vec{0}$  and  $\frac{d\hat{j}}{dt} = \vec{0}$ ). Through differentiation with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} \qquad \dot{x} = v_x, \quad \dot{y} = v_y$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{i} + \ddot{y}\hat{j} \qquad \ddot{x} = a_x, \quad \ddot{y} = a_y$$

## Discussion – Cartesian Description

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

From these equations, we see that the determination of the velocity and acceleration of a point in Cartesian components depends on our ability to differentiate the  $x$  and  $y$  components of its position with respect to time. Let's focus our attention to the difference between the EXPLICIT and IMPLICIT time dependence of these Cartesian components:

- If  $x$  and  $y$  are explicit functions of time,  $x(t)$  and  $y(t)$ , then the Cartesian components for velocity and acceleration vectors are found directly by time differentiation of these functions.
- If the path of the point is given by the function  $y = f(x)$ , for example, and the kinematics are known for  $x(t)$  ( $y$  is an implicit function of time;  $x$  is an explicit function of time), then we have to use the chain rule of differentiation. In this particular case, we have:

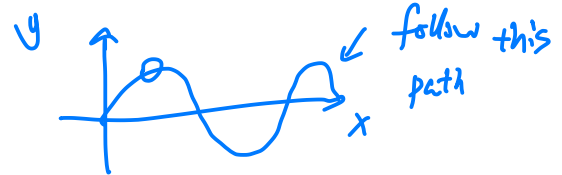
$$\dot{y} = \frac{dy}{dt} = \frac{d}{dt}f(x) = \frac{df}{dx} \frac{dx}{dt} = \dot{x} \frac{df}{dx}$$

Consider the following example:

**MOTIVATING EXAMPLE** Suppose that  $y = \sin x$  and  $\dot{x} = 3 \text{ m/s} = \text{constant}$ , and we want to know the velocity and acceleration when  $x = \pi/2$ . The Cartesian components of velocity and acceleration are given by:

$$\begin{aligned} \dot{y} &= \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \\ &= \dot{x} \frac{d}{dx}(\sin x) = \dot{x} \cos x \Rightarrow \dot{y} = (3) \cos \frac{\pi}{2} = 0 \text{ m/s} \end{aligned}$$

$$\begin{aligned} \ddot{y} &= \frac{d}{dt}(\dot{x} \cos x) = \ddot{x} \cos x + \dot{x}(-\dot{x} \sin x) \\ &= \ddot{x} \cos x - \dot{x}^2 \sin x \Rightarrow \ddot{y} = (0) \cos \frac{\pi}{2} - (3)^2 \sin \frac{\pi}{2} = -9 \text{ m/s}^2 \end{aligned}$$



Therefore,

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = (3)\hat{i} + (0)\hat{j} = 3\hat{i} \text{ m/s}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (0)\hat{i} + (-9)\hat{j} = -9\hat{j} \text{ m/s}^2$$

Note: ① always check which variable is a function of  $t$

② always work out expression before

Substitute numbers.

**Example 1.A.1**

**Given:** Pin P is constrained to move along an elliptical ring whose shape is given by  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  (where  $x$  and  $y$  are given in mm). The pin is also constrained to move within a horizontal slot that is moving upward at a constant speed of  $v$ .

**Find:** Determine:

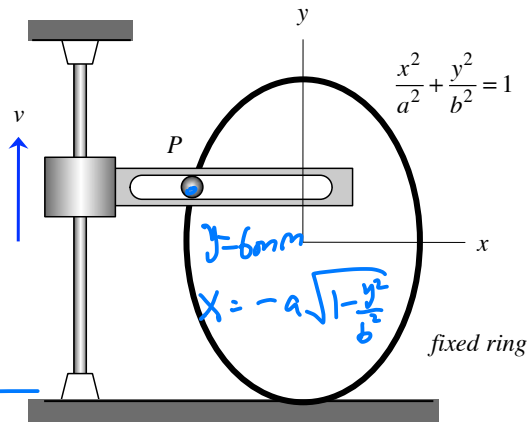
- (a) The velocity of pin P at the position corresponding to  $y = 6$  mm; and
- (b) The acceleration of pin P at the position corresponding to  $y = 6$  mm.

Use the following parameters in your analysis:  $a = 5$  mm,  $b = 10$  mm,  $v = 30$  mm/s.

$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$ ,  $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

$\dot{y} = v - \text{Const.}$ ,  $\ddot{y} = 0$

$x = f(y)$ ,  $\dot{x} = \frac{df}{dy} \cdot \dot{y}$  ...  
- Complicated



We can perform  $\frac{d}{dt}$  to a equation

(a)  $\frac{d}{dt} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = 0$

$\frac{2x\dot{x}}{a^2} + \frac{2y\dot{y}}{b^2} = 0 \Rightarrow \dot{x} = -\frac{a^2}{x} \cdot \frac{\dot{y}}{b^2} \rightarrow \text{Substitute.}$

$\dot{y} = 6 \text{ mm}$ ,  
 $\dot{y} = 30 \text{ mm/s}$   
 $x = -a \sqrt{1 - \frac{y^2}{b^2}}$

(b)  $\frac{d}{dt} \left( \frac{2x\dot{x}}{a^2} + \frac{2y\dot{y}}{b^2} \right) = 0$

$\frac{2(\dot{x}\dot{x} + x\ddot{x})}{a^2} + \frac{2(\dot{y}\dot{y} + y\ddot{y})}{b^2} = 0 \Rightarrow \ddot{x} = -\frac{a^2}{x} \left( \frac{\dot{y}^2}{b^2} + \frac{\ddot{y}}{a^2} \right)$

Substitute:  $\dot{y} = v$ ,  $\ddot{y} = 0$ ,  $\dot{x}$  from (a)