Moving Reference Frame Kinematics

Background

In the preceding lectures, we saw that if two points, A and B, are on the SAME rigid body that is undergoing planar motion, their velocities and accelerations are related by the following equations:

\[ \vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \]

\[ \vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \]

It is often the case that we need to relate the motion of two points that are NOT on the same rigid body. In this case, the above kinematic equations are not valid. So what equations can we use?

Many times we can use kinematic information about the motion of B obtained from an observer on a “moving reference frame” (“moving” includes both translation and rotation). This is shown in the figure to the right where an observer on a moving reference frame is watching point B. Let’s say that a set of \( x, y, z \) coordinate axes are attached to this reference frame. In this case, the observer at A would describe the position of B as:

\[ \vec{r}_{B/A} = x_{B/A} \hat{x} + y_{B/A} \hat{y} \]

Note that in this description, the unit vectors \( \hat{i} \) and \( \hat{j} \) have time-varying orientations (and therefore are not constant vectors) since they are rotating with the moving reference frame.

Objectives

The goals of this chapter are to:

- develop the “rotating reference frame” velocity and acceleration kinematic equations
- apply these equations to the analysis of more complicated planar mechanisms and to problems in three dimensions
A. Kinematics - 2D Rotating Reference Frames

Angular Velocity of the Rotating Reference Frame – 2D

Let’s start first with the concept of the “angular velocity”, \( \vec{\omega} \), of the observer for planar motion. As stated before, the observer is attached to the moving (translating and rotating) \( xyz \) coordinate axes. In addition, we also introduce a set of stationary \( XYZ \) axes, as shown in the figure to the right. The angle of rotation \( \theta \) shown in the figure is the angle between the \( X \)- and \( x \)-axes, as well as the angle between the \( Y \)- and \( y \)-axes. The angular speed of the observer is given by: 

\[ \vec{\omega} = \dot{\theta} \hat{k} \]

Although the unit vectors \( \hat{i} \) and \( \hat{j} \) are of constant length, they change in their direction as the observer rotates in space. Therefore, their time derivatives are not zero. From the equations in the figure to the right we see that

\[
\frac{d\hat{i}}{dt} = \dot{\theta} \hat{j} \\
\frac{d\hat{j}}{dt} = -\dot{\theta} \hat{i}
\]

From the right-hand rule, we know that:

\[ \hat{i} = -\hat{k} \times \hat{j} \]
\[ \hat{j} = \hat{k} \times \hat{i} \]

Therefore,

\[
\frac{d\hat{i}}{dt} = \dot{\theta} (\hat{k} \times \hat{i}) = \dot{\theta} \hat{k} \times \hat{i} = \vec{\omega} \times \hat{i} \\
\frac{d\hat{j}}{dt} = -\dot{\theta} (\hat{k} \times \hat{j}) = \dot{\theta} \hat{k} \times \hat{j} = \vec{\omega} \times \hat{j}
\]

These equations will be useful to us in deriving our moving reference frame kinematics equations.
Velocity Equation – 2D

The position of point B can be written in terms of the position of point A as:

\[ \vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \]

where \( \vec{r}_{B/A} \) can be written in terms of the observer’s \( xy \) coordinates as:

\[ \vec{r}_{B/A} = x\hat{i} + y\hat{j} \]

Differentiating this expression with respect to time gives (remembering that \( \hat{i} \) and \( \hat{j} \) are NOT constant vectors):

\[ \frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \quad \implies \]

\[ \vec{v}_B = \vec{v}_A + \frac{d}{dt} (x\hat{i} + y\hat{j}) \]

\[ = \vec{v}_A + x \frac{dx}{dt} \hat{i} + x \frac{dy}{dt} \hat{j} + y \frac{dx}{dt} \hat{j} + y \frac{dy}{dt} \hat{j} \]

\[ = \vec{v}_A + \vec{a}_B + \vec{\omega} \times \vec{r}_{B/A} \]

where

\[ (\vec{v}_{B/A})_{\text{rel}} = \hat{x}\omega + \hat{y}\omega = \text{“velocity of B as seen by the moving observer”} \]

Recall in Chapter 1: \( \vec{V}_b' = \vec{V}_h + \vec{V}_{B/H} \)

\[ \Downarrow \]

relative velocity

\[ \Rightarrow \vec{V}_{B/H} = (\vec{V}_{B/H})_{\text{rel}} + \vec{\omega} \times \vec{r}_{B/H} \]

observed velocity ≠ relative velocity
Acceleration Equation – 2D

In the preceding section we saw that the velocity of B can be written as:

\[ \vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A} \]

where \( \vec{\omega} \) is the angular velocity of the observer at A and \( (\vec{v}_{B/A})_{rel} \) is the “velocity of B as seen by the observer at A”. We have also seen that:

\[ \frac{d}{dt} \hat{i} = \vec{\omega} \times \hat{i} \]
\[ \frac{d}{dt} \hat{j} = \vec{\omega} \times \hat{j} \]

Differentiating the above with respect to time gives:

\[ \frac{d}{dt} \vec{v}_B = \frac{d}{dt} \vec{v}_A + \frac{d}{dt} (\vec{v}_{B/A})_{rel} + \frac{d}{dt} (\vec{\omega} \times \vec{r}_{B/A}) \quad \Rightarrow \quad \begin{array}{c} \text{usual product rule for } \frac{d}{dt} \\ \text{also applies to cross product} \end{array} \]

\[ \vec{a}_B = \vec{a}_A + \frac{d}{dt} (\vec{v}_{B/A})_{rel} + \frac{d}{dt} (\vec{\omega} \times \vec{r}_{B/A}) + \vec{\omega} \times \left[ \frac{d}{dt} \left( \vec{v}_{B/A} \right)_{rel} + \vec{\omega} \times \vec{r}_{B/A} \right] \]
\[ = \vec{a}_A + \frac{d}{dt} \left( \hat{x} \hat{i} + \hat{y} \hat{j} \right) + \frac{d}{dt} \left( \hat{x} \frac{d}{dt} \hat{i} + \hat{y} \frac{d}{dt} \hat{j} \right) + \frac{d}{dt} (\vec{\omega} \times \vec{r}_{B/A}) + \vec{\omega} \times \left[ \frac{d}{dt} \left( \vec{v}_{B/A} \right)_{rel} + \vec{\omega} \times \vec{r}_{B/A} \right] \]
\[ = \vec{a}_A + \frac{d}{dt} \left( \hat{x} \hat{i} + \hat{y} \hat{j} \right) + \frac{d}{dt} \left( \vec{\omega} \times \hat{i} + \vec{\omega} \times \hat{j} \right) + \frac{d}{dt} \left( \vec{v}_{B/A} \right)_{rel} + \vec{\omega} \times \left[ \frac{d}{dt} \left( \vec{v}_{B/A} \right)_{rel} + \vec{\omega} \times \vec{r}_{B/A} \right] \]
\[ = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A} + 2 \vec{\omega} \times (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \left[ \vec{\omega} \times \vec{r}_{B/A} \right] \]

where

\[ (\vec{a}_{B/A}) = \hat{x} \hat{i} + \hat{y} \hat{j} \quad \text{“acceleration of B as seen by the moving observer”} \]
\[ \vec{\alpha} = \text{“angular acceleration of the observer”} = \frac{d}{dt} \vec{\omega}. \]
Product rule for vector derivatives

1. If $r_1(t)$ and $r_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

**Answer:** As with the dot product, this will follow from the usual product rule in single variable calculus. We want to show

$$\frac{d(r_1 \times r_2)}{dt} = r'_1 \times r_2 + r_1 \times r'_2.$$ 

Let $r_1 = \langle x_1, y_1, z_1 \rangle$ and $r_2 = \langle x_2, y_2, z_2 \rangle$. We have,

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle.$$ 

Taking derivatives using the product rule from single variable calculus, we get a lot of terms, which we can group to prove the vector formula.

$$\frac{d(r_1 \times r_2)}{dt} = \langle y'_1 z_2 + y_1 z'_2 - z'_1 y_2 - z_1 y'_2, z'_1 x_2 + z_1 x'_2 - x'_1 z_2 - x_1 z'_2, x'_1 y_2 + x_1 y'_2 - y'_1 x_2 - y_1 x'_2 \rangle$$

$$= \langle (y'_1 z_2 - z'_1 y_2) + (y_1 z'_2 - z_1 y'_2), (z'_1 x_2 - x'_1 z_2) + (z_1 x'_2 - x_1 z'_2), (x'_1 y_2 - y'_1 x_2) + (x_1 y'_2 - y_1 x'_2) \rangle$$

$$= \langle x'_1, y'_1, z'_1 \rangle \times \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle \times \langle x'_2, y'_2, z'_2 \rangle$$

$$= r'_1 \times r_2 + r_1 \times r'_2. \quad \blacksquare$$
Discussion - What Do Each of the Terms Represent?

\[ \vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{\text{rel}} + \vec{\omega} \times \vec{r}_{B/A} \]
\[ = \vec{v}_A + \vec{v}_{B/A} \]
\[ \vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{\text{rel}} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times (\vec{v}_{B/A})_{\text{rel}} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] \]
\[ = \vec{a}_A + \vec{a}_{B/A} \]

- \( \vec{v}_A \) and \( \vec{v}_B \) are the velocities of points B and A as seen by a fixed observer.
- \( \vec{a}_A \) and \( \vec{a}_B \) are the accelerations of points B and A as seen by a fixed observer.
- \( \vec{\omega} \) is the angular velocity of the moving observer.
- \( \vec{\alpha} \) is the angular acceleration of the moving observer.
- \( (\vec{v}_{B/A})_{\text{rel}} \) is the “velocity of point B as seen by the moving observer at A”. In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- \( (\vec{a}_{B/A})_{\text{rel}} \) is the “acceleration of point B as seen by the moving observer at A”. In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- The term \( 2\vec{\omega} \times (\vec{v}_{B/A})_{\text{rel}} \) is known as the “Coriolis” component of acceleration. The term arises due to an observed velocity of B by the moving observer, when the observer has a non-zero angular velocity. We will introduce a number of physical examples that demonstrate the significance of this term during lecture.

**CHALLENGE QUESTIONS:** What are the differences between \( \vec{v}_{B/A} \) and \( (\vec{v}_{B/A})_{\text{rel}} \)? And between \( \vec{a}_{B/A} \) and \( (\vec{a}_{B/A})_{\text{rel}} \)? Are they ever the same?

**ANSWERS:** Physically, we know that \( (\vec{v}_{B/A})_{\text{rel}} \) and \( (\vec{a}_{B/A})_{\text{rel}} \) are the velocity of B and the acceleration of B as seen by the moving observer, respectively. Mathematically, from our earlier derivations, we know:

\[ (\vec{v}_{B/A})_{\text{rel}} = \vec{v}_{B/A} - \vec{\omega} \times \vec{r}_{B/A} \]
\[ (\vec{a}_{B/A})_{\text{rel}} = \vec{a}_{B/A} - \vec{\alpha} \times \vec{r}_{B/A} - 2\vec{\omega} \times (\vec{v}_{B/A})_{\text{rel}} - \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] \]

From this, we see that if the observer is translating but not rotating (i.e., \( \vec{\omega} = \vec{\alpha} = \vec{0} \)), \( (\vec{v}_{B/A})_{\text{rel}} = \vec{v}_{B/A} \) and \( (\vec{a}_{B/A})_{\text{rel}} = \vec{a}_{B/A} \). The rotational terms on the right hand side of the above equations provide us with information on how the motion of B as seen by the observer is distorted by the rotation of the observer.
Discussion – The “How To” with Moving Reference Frame Kinematics Equations

1. Choose your moving reference frame (observer). It is recommended that you draw a stick figure of your observer on this frame to remind yourself of your choice of reference frame. Note that point A must be on this reference frame.

2. Draw your choice of $xyz$ axes for the moving reference frame. State in words to what the $xyz$ axes are attached. Also show your choice of stationary $XYZ$ axes for the problem.

3. Determine the angular velocity $\vec{\omega}$ of the moving reference frame. [Note: this represents the ANGULAR MOTION OF THE OBSERVER, and not what the observer sees.]

4. Determine the angular acceleration $\vec{\alpha}$ of the moving reference frame.

5. Imagine yourself as the observer on the moving reference frame. Answer the question: How do I see point B move if I am that observer? Based on this answer, write down $\left( \vec{v}_{B/A} \right)_{rel}$ and $\left( \vec{a}_{B/A} \right)_{rel}$. [Note: this is the MOTION THAT THE OBSERVER SEES, not the motion of the observer.]

6. In the last chapter of the lecture notes, we studied the planar kinematics of mechanisms made up of rigid links connected by pin joints. For these problems, we typically wrote down rigid body kinematics equations relating the motion of the pins within the mechanism. Consider the mechanism shown below where pin A slides within a slot cut in line BC. For this problem, we cannot use rigid body kinematics equations relating the motion of pin A to pin C. Why? (The distance between A and C is not fixed, as required by the rigid body kinematics equations.). How can we go about solving this problem? It is recommended that you use a moving reference frame equation to relate the motion of A and C. More specifically, for velocity analysis, use the following rigid body equation for link OA:

$$\vec{v}_A = \vec{v}_O + \vec{\omega}_{OA} \times \vec{r}_{A/O} = \vec{\omega}_{OA} \times \vec{r}_{A/O}$$
and the following moving reference equation for link BC (employing an observer on link BC):

$\vec{v}_A = \vec{v}_O + (\vec{v}_{A/C})_{rel} + \vec{\omega}_{BC} \times \vec{r}_{A/C} = \dot{L} \hat{j} + \vec{\omega}_{BC} \times \vec{r}_{A/C}$

With $\omega_{BC}$ known, equating $\vec{v}_A$ from the above two equations produces two scalar equations ($\hat{i}$ and $\hat{j}$ components) in terms of two unknowns: $\omega_{OA}$ and $\dot{L}$. You will use this solution process in a number of examples and homework problems related to this section of the notes.
Example 3.A.1

**Given:** The disk shown below rolls without slipping on a horizontal surface. At the instant shown, the center O is moving to the right with a speed of \( v_0 = 5 \text{ m/s} \) with this speed decreasing at a rate of \( 2 \text{ m/s}^2 \). Also for this instant, the particle P is at a position of \( x_p = 0.2 \text{ m} \) with \( \dot{x}_p = 2 \text{ m/s} = \text{constant} \), where \( x_p \) is measured relative to the \( xyz \) coordinate system that is attached to the disk.

**Find:** Determine:
(a) The velocity of particle P; and  
(b) The acceleration of particle P.

Use the following parameters in your analysis: \( h = 0.2 \text{ m} \) and \( r = 0.6 \text{ m} \).

![Diagram of disk and particle P](image)

\[ \dot{\omega} = -\frac{v_0}{r} \hat{z} \]
\[ \vec{v}_p = \vec{v}_0 + (\dot{\vec{v}}_p) + \vec{\omega} \times \vec{r}_p \]
\[ = v_0 \hat{i} + \dot{x}_p \hat{i} + (-\frac{v_0}{r} \hat{z}) \times (x_p \hat{i} + h \hat{j}) \]
\[ \text{at this instance: } \hat{i} = \hat{i}, \hat{j} = \hat{j}, \hat{k} = \hat{z}, \text{ but in general: } \hat{i} = \hat{k}, \hat{j} = \hat{k}, \hat{k} = \hat{k} \]
\[ = u \hat{i} + \dot{x}_p \hat{i} - \frac{v_0}{r} x_p \hat{j} + \frac{v_0}{r} h \hat{j} \]
\[ = (u + \dot{x}_p + \frac{v_0}{r} h) \hat{i} - \frac{v_0}{r} x_p \hat{j} \]

\[ \vec{a}_p = \vec{a}_0 + (C\vec{v}_p)_x + \vec{a}_w + \vec{a}_r \]

We need \( \vec{a}_r \):
\[ \vec{a}_0 = \vec{a}_c + \vec{a}_w + \vec{a}_r \]
\[ = a_c \hat{c} - 2 r \hat{r} - \omega^2 \hat{r} \]
\[ \vec{a}_r = -\frac{a_c}{r} \]

\[ \frac{v_0}{r} = \text{constant} \]
At this instance $I = \dot{I}$, $j = j'$, $k = k'$

\[
\dot{\gamma} = a_0 \dot{I} + (-\frac{a_0 I}{v}) \times (x_p \dot{r} + \dot{h}^2) + 2 \left( -\frac{v_0^2}{r^2} \right) \times \dot{x}_p \dot{r} - \frac{w^2}{r^2} \left( x_p \dot{r} + \dot{h}^2 \right)
\]

\[
= a_0 \dot{I} - \frac{a_0 x_p}{r} \dot{r} + \frac{a_0 h}{r} \dot{r} - \frac{2w_0 x_p}{r} \dot{r} - \frac{w_0^2}{r^2} \dot{x}_p \dot{r} - \frac{w_0^2}{r^2} \dot{h}^2
\]

\[
= \left( a_0 \dot{I} + \frac{a_0 h}{r} \dot{r} - \frac{w_0^2}{r^2} \dot{x}_p \dot{r} \right) \dot{r} + \left[ -\frac{a_0 x_p}{r} - \frac{2w_0 x_p}{r} - \frac{w_0^2}{r^2} \right] \dot{x}_p \dot{r}
\]