C. Summary: Kinematic Analysis of Planar Mechanisms

The ONLY equations that you will need for the kinematic analysis of the planar motion of rigid bodies in a mechanism are the following:

\[ \vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \]
\[ \vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \]

Consider the following points as you review the following worked-out kinematics problem for a planar mechanism.

**Reminders for solving the velocity and acceleration problems:**

1. Define your set of coordinate axes. The following discussion will assume that the coordinate axes have been drawn such that the \( \hat{k} \) axis direction points out of the paper (be sure to check this using the right-hand rule).

2. Write down a velocity (acceleration) equation for EACH rigid link in the mechanism by relating the motion of two points on each link.
   - Assume that unknown angular velocity (angular acceleration) terms point in the \( +\hat{k} \) direction. If you get a negative sign in the final answer, this just means that you made the wrong assumption. *Given* angular velocities must be given the correct sign (CCW = + and CW = -).
   - Recall that the position vector \( \vec{r}_{B/A} \) points FROM A TO B.
   - Points A and B must be on the same rigid body.
   - Be sure to give each angular velocity (angular acceleration) a unique subscript. Leaving off subscripts on these terms will lead to confusion in later steps.

3. Combine your equations from step 2 above.
   - Count the number of scalar equations. Note that each vector equation has two scalar components (\( x \) and \( y \) components).
   - Count the number of unknowns.
   - If you have enough equations for the number of unknowns, balance the coefficients of \( \hat{i} \) and \( \hat{j} \). Solve these equations.
   - If you do not have enough equations, then go back to step 2 to see which equations that you have missed. After arriving at having enough equations, solve for the unknowns as above.
   - Write your answers as vectors. Note that angular velocities (and angular accelerations) are in the \( \hat{k} \) direction. The sign on these vectors depend on the sign found in your solution above.

4. Use the concept of instant centers to check the direction of the velocity and angular velocity vectors in your answers.
Example 2.C.1

**Given:** The wheel rolls without slipping in such a way that slider B moves to the left with a constant speed of $v_B = 5 \text{ ft/s}$.

**Find:** Determine:

(a) The angular velocity of the wheel when $\theta = 0$; and

(b) The angular acceleration of the wheel when $\theta = 0$.

Use the following parameters in your analysis: $L = 2 \text{ ft}$ and $r = 0.5 \text{ ft}$.

**Wheel:**

\[ \vec{V}_A = \vec{V}_C + \omega_{BA} \times \vec{r}_{AB} = \omega_{BA} \times \left[ r \cos \theta \mathbf{i} + r (1 + \sin \theta) \mathbf{j} \right] \]

\[ = -\omega_{BA} r (1 + \sin \theta) \mathbf{i} + \omega_{BA} r \cos \theta \mathbf{j} \quad \text{(1)} \]

**Link AB:**

\[ \vec{V}_B = \vec{V}_B + \omega_{AB} \times \vec{r}_{AB} = -\vec{V}_B + \omega_{AB} \times (-L_x \mathbf{i} + L_y \mathbf{j}) \]

\[ = (-\vec{V}_B - \omega_{AB} L_y) \mathbf{i} - \omega_{AB} L_x \mathbf{j} \quad \text{(2)} \]

\[ \text{Equating (1) and (2)} \Rightarrow \begin{cases} -\omega_{BA} r (1 + \sin \theta) = -\vec{V}_B - \omega_{AB} L_y \\ \omega_{BA} r \cos \theta = -\omega_{AB} L_x \end{cases} \]

\[ \Rightarrow \omega_{BA} = \frac{\vec{V}_B L_x + \omega_{AB} L_y}{L_y} \quad \omega_{BA} = \frac{-\vec{V}_B}{L_x + L_y} \]
**Wheel:***

\[
\vec{a}_w = \vec{a}_c + \vec{\omega}_{BA} \times \vec{r}_{AC} + \vec{\omega}_{BA} \times (\vec{\omega}_{BA} \times \vec{r}_{AC/3})
\]

\[
= \vec{w}_A \times r_i + 2 \vec{\omega}_{BA} \times \left[ r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right]
\]

\[
+ \vec{w}_{AB} \times \left[ \vec{w}_{AB} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right]
\]

\[
= \vec{w}_A \times r_i + 2 \vec{\omega}_{BA} \times \left[ r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right]
\]

\[
- \vec{w}_{AB} \times \left[ \vec{w}_{AB} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right]
\]

\[
= \left( -2 \vec{\omega}_{BA} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right) \cdot \hat{i}
\]

\[
+ \vec{w}_{AB} \times \left[ \vec{w}_{AB} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right] \cdot \hat{j}
\]

\[
= \left[ -2 \vec{\omega}_{BA} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right] \cdot \hat{i}
\]

\[
+ \left[ \vec{w}_{AB} \times \left( r \cos \theta \hat{i} + r(1+\sin \theta) \hat{j} \right) \right] \cdot \hat{j}
\]

**Link AB:**

\[
\vec{a}_B = \vec{a}_B + \vec{\omega}_{AB} \times \vec{r}_{AB/3} + \vec{w}_{AB} \times (\vec{\omega}_{AB} \times \vec{r}_{AB/3})
\]

\[
= \vec{2} \vec{\omega}_{AB} \times \left( -L_x \hat{i} + L_y \hat{j} \right) + \vec{w}_{AB} \times \left[ \vec{w}_{AB} \times \left( -L_x \hat{i} + L_y \hat{j} \right) \right]
\]

\[
= -2 \vec{\omega}_{AB} \times \hat{i} + \vec{w}_{AB} \times \hat{j} - 2 \vec{\omega}_{AB} \times \hat{j}
\]

\[
= \left( -2 \vec{\omega}_{AB} \times \hat{j} + \vec{w}_{AB} \right) \hat{i} + \left( -2 \vec{\omega}_{AB} \times \hat{i} - \vec{w}_{AB} \right) \hat{j}
\]

\[
= \left( -2 \vec{\omega}_{AB} \times \hat{j} + \vec{w}_{AB} \right) \hat{i} + \left( -2 \vec{\omega}_{AB} \times \hat{i} - \vec{w}_{AB} \right) \hat{j}
\]

\[
\Rightarrow \quad \boxed{2 \vec{\omega}_{AB} \times \hat{j} + \vec{w}_{AB} \hat{i} + \left( -2 \vec{\omega}_{AB} \times \hat{i} - \vec{w}_{AB} \right) \hat{j}
\]

\[
\Rightarrow \quad \boxed{\vec{\omega}_{AB} = -\frac{L_x}{2} \hat{i} - \frac{L_y}{2} \hat{j}}
\]

**Notes:**

- wheel and roller slides in the direction of motion.
- \( \vec{a}_w \) and \( \vec{a}_B \) are acceleration vectors.
- \( \vec{\omega}_{BA} \) and \( \vec{\omega}_{AB} \) are angular velocity vectors.
- \( \vec{w}_{AB} \) is the tangential velocity vector.
- \( \vec{r}_{AC/3} \) is the vector from point A to point C.
Example 2.C.2

**Given:** Link OA, of length \( L \), in the mechanism shown below has a constant counterclockwise angular velocity of \( \omega_{OA} \) as it moves into a vertical position. At this same instant, link BD is also vertically oriented, where pins O and D lie along the same horizontal line.

**Find:** Determine, at this instant in time:
(a) The angular velocity of link AB;
(b) The angular velocity of link BD;
(c) The angular acceleration of link AB; and
(d) The angular acceleration of link BD

Use the following parameter in your analysis: \( \omega_{OA} = 0.5 \) rad/s.

\[ \begin{align*}
(a) & \quad \text{O is IC & OA } \Rightarrow \overrightarrow{v}_A = -\omega_{OA} L \hat{k} \\
& \quad \text{D is IC & BD } \Rightarrow \overrightarrow{v}_B = v_B \hat{k}
\end{align*} \]

\[ \begin{align*}
\overrightarrow{v}_A \parallel \overrightarrow{v}_B & \quad \Rightarrow \overrightarrow{w}_{AB} = 0 \\
\overrightarrow{v}_B \perp \overrightarrow{v}_A & \quad \Rightarrow \overrightarrow{v}_B = \overrightarrow{v}_B = -\omega_{OA} L \hat{k} \\
\Rightarrow \quad \overrightarrow{w}_{BD} = \frac{\omega_{OA}}{2} \hat{k}
\end{align*} \]

\[ \text{(b) } \quad |\overrightarrow{w}_{BD}| = \frac{|\overrightarrow{v}_B|}{2L} = \frac{\omega_{OA}}{2}, \text{ CCW} \]

\[ \Rightarrow \quad \overrightarrow{w}_{BD} = \frac{\omega_{OA}}{2} \hat{k} \]

\[ \text{(c) Link OA: } \overrightarrow{a}_A = \overrightarrow{a}_0 + \overrightarrow{a}_{OA} \times \overrightarrow{v}_{A/0} + \overrightarrow{w}_{oh} \times (\overrightarrow{v}_{oh} \times \overrightarrow{v}_{A/0}) = -\omega_{OA}^2 L \hat{j} \]

\[ \text{Link BD: } \overrightarrow{a}_B = \overrightarrow{a}_0 + \overrightarrow{a}_{BD} \times \overrightarrow{v}_{B/0} + \overrightarrow{w}_{BD} \times (\overrightarrow{v}_{BD} \times \overrightarrow{v}_{B/0}) \]

\[ \Rightarrow \quad = -2L \omega_{BD} \hat{i} - 2\omega_{BD}^2 \hat{j} \]
Link AB: \[ \mathbf{A}_b = \mathbf{a} + \mathbf{a}_{AB} \times r_{13/A} + \mathbf{w}_{AB} \times (\mathbf{w}_{AB} \times r_{13/A}) \]
\[ = -w_{ab} \mathbf{L}_j \]

\[ \text{from eq. 1} \]
\[ = -w_{ab} \mathbf{L}_j + \mathbf{a}_{AB} \times (2 \mathbf{L}_i + \mathbf{L}_j) \]
\[ = -\mathbf{a}_{AB} \mathbf{L}_i + (-w_{ab} \mathbf{L} + 2 \mathbf{a}_{AB} \mathbf{L}) \]

\[ \mathbf{2} = \mathbf{3} \implies \begin{cases} -2L \mathbf{a}_{BD} = -\mathbf{a}_{AB} \mathbf{L} \\ -2w_{BD} \mathbf{L} = -w_{ab} \mathbf{L} + 2 \mathbf{a}_{AB} \mathbf{L} \end{cases} \]

\[ \Rightarrow \begin{cases} \mathbf{a}_{AB} = \frac{w_{ab}^2}{2} - \mathbf{a}_{BD} \\ \mathbf{a}_{BD} = \frac{w_{ab}^2}{4} - \frac{w_{BD}^2}{2} \end{cases} \]
Question C2.2

A sphere of radius \( r \) rolls without slipping to the right on a rough, horizontal surface. The center of the sphere, O, has a speed of \( v_O \), with this speed decreasing at a rate of \( a_O \).

Circle the figure below that most accurately represents the direction of the acceleration of the contact point C.

- **Figure A**
- **Figure B**
- **Figure C**
- **Figure D**
- **Figure E**
- **Figure F**
- **Figure G**
- **Figure H**
- **Figure I**
Question C2.3
A stepped drum has inner and outer radii of $R$ and $2R$, respectively. The drum rolls to the left with its center O having a constant speed of $v_O$, as shown below. Point A and B lie on the inner and outer radii, respectively, of the drum. At the instant shown, A is directly to the right of O, and B is directly below O. For this position:

(a) Make a sketch of the velocity vectors for A and B.
(b) Make a sketch of the acceleration vectors for A and B.
Question C2.7
The mechanism shown below is made up on rigid links OA, AB and BE. Link OA is rotating in the counterclockwise direction with an angular speed of \( \omega_{OA} \).

(a) Accurately locate the instant center for link AB.

(b) Is AB rotating clockwise, rotating counterclockwise or instantaneously stationary? Justify your answer.

(c) Is BE rotating clockwise, rotating counterclockwise or instantaneously stationary? Justify your answer.