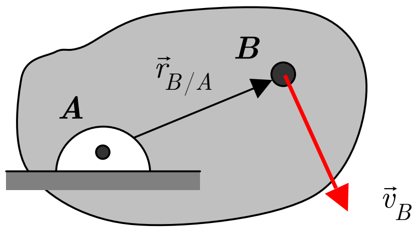
B. Instantaneous Centers of Rotation (Instant Centers)

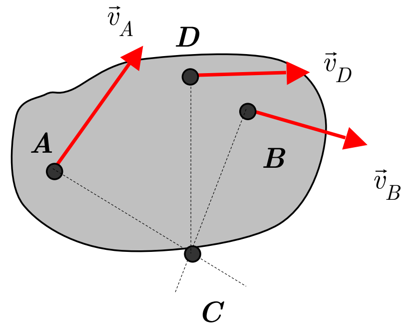
CHALLENGE QUESTION: Suppose we consider a rigid body that is pinned to ground at point A ($\vec{v}_A = \vec{0}$). We say that the body is *rotating about point A* (or, equivalently, A is the center of rotation). What is the direction and magnitude of the velocity of some point B on the body?



ANSWER: The velocity of B, $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$, is perpendicular to the line connecting B with the center of rotation A. The speed of B is given by $v_B = \omega \left|\vec{r}_{B/A}\right|$.

given $\vec{V}$, how to find center of rotation?

CHALLENGE QUESTIONS: Consider another rigid body where we know the direction of the velocity vectors for two points A and B on the body for a given instant in time. Where is the center of rotation for this instant (the *instantaneous center of rotation*)? How can we use the location of this center of rotation to determine the velocity of a third point D?



ANSWERS:

- The center of rotation must be on a line that is perpendicular to $\vec{v}_A$. The center of rotation must also be on a line that is perpendicular to $\vec{v}_B$. Therefore, the center of rotation C is at the intersection of these two perpendiculars.

Does $\vec{r}_{C/A} \perp \vec{v}_A$ and $\vec{r}_{C/B} \perp \vec{v}_B$ __Always__ guarantee $\vec{v}_C = 0$?

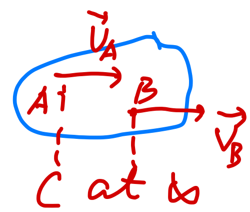
2D:

$$\left.\begin{array}{l} \vec{r}_{C/A} \perp \vec{v}_A \Rightarrow \vec{v}_C = \vec{v}_A + \omega\hat{k} \times \vec{r}_{C/A} \\ \omega\hat{k} \times \vec{r}_{C/A} \parallel \vec{v}_A \end{array}\right\} \Rightarrow \vec{v}_C \parallel \vec{v}_A$$

similarly: $\vec{r}_{C/B} \perp \vec{v}_B \Rightarrow \vec{v}_C \parallel \vec{v}_B$

Case 1. $\vec{v}_A \not\parallel \vec{v}_B \Rightarrow \vec{v}_C = \vec{0}$

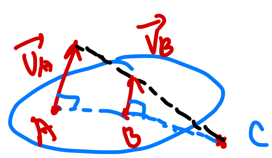
Case 2. $\vec{v}_A \parallel \vec{v}_B$, $\vec{r}_{B/A}$ NOT $\perp \vec{v}_A$



$$\Rightarrow \quad \left.\begin{array}{l} \vec{v}_B = \vec{v}_A + \underbrace{\omega\hat{k} \times \vec{r}_{B/A}}_{\text{not} \parallel \vec{v}_A} \\ \vec{v}_B \parallel \vec{v}_A \end{array}\right\} \Rightarrow \underline{\omega = 0} \Rightarrow \vec{v}_A = \vec{v}_B$$

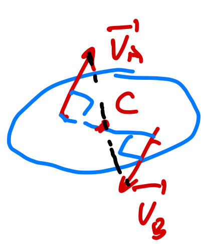
$\longrightarrow$ Rigid body translation

Case 3.



$$\Rightarrow \quad \omega = \frac{|\vec{v}_B|}{|\vec{r}_{B/C}|} = \frac{|\vec{v}_A|}{|\vec{r}_{A/C}|}$$

Case 4.



$$\Rightarrow \quad \omega = \frac{|\vec{v}_A|}{|\vec{r}_{C/A}|} = \frac{|\vec{v}_B|}{|\vec{r}_{C/B}|}$$

- The velocity of ANY point on the body (D) must be perpendicular to the line connecting C and D.
- Point C is known as the "instantaneous center of rotation" (or simply, the "instant center") of the body.

What we have seen from our answers above can be summarized as the following:

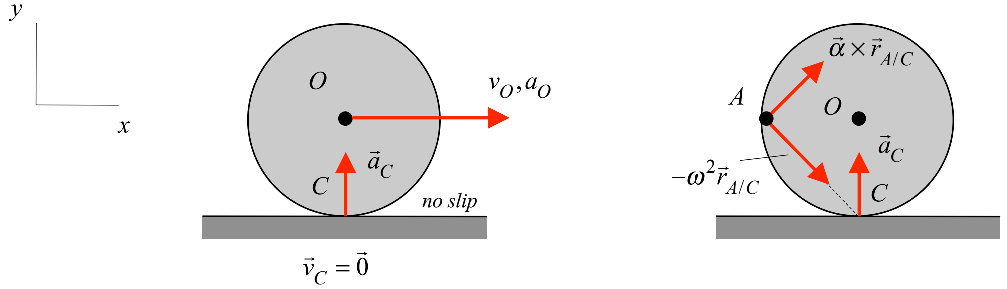
- The instantaneous center of rotation for a body is located on a line perpendicular to the velocity vector for any point on the body. The intersection of these perpendiculars provide us with the actual location of this instantaneous center of rotation C.
- Since C is the center of rotation, $\vec{v}_C = \vec{0}$. Therefore, the velocity of any point on the body is perpendicular to the line connecting C to that point.
- The speed of any point is proportional to the distance from that point to C. As examples, for points A, B and D shown in the above figure: $v_A = \omega|\vec{r}_{A/C}|$, $v_B = \omega|\vec{r}_{B/C}|$ and $v_D = \omega|\vec{r}_{D/C}|$. Alternately, we can write these as: $\omega = v_A/|\vec{r}_{A/C}| = v_B/|\vec{r}_{B/C}| = v_D/|\vec{r}_{D/C}|$.
- QUESTION: What does it say about ω if we find that $|\vec{r}_{A/C}| = \infty$ (if the perpendicular lines intersect at ∞)? $\omega = 0$, Case 2. above
- The sense of rotation (that is, the sign of ω) is determined by visualizing the rotation of the body about C. For the example shown above, the body is rotating with a clockwise sense about point C. Clockwise or Counter clockwise
- We call point C the "instant center of rotation". When moving to a new position, the directions of the velocities for the points of the body will change, and, as result, the location of C will change. Note that the instant center C does not need to physically lie on the body itself since we are talking about the point about which the body is rotating.

CHALLENGE QUESTIONS: The instant center of rotation for a body has zero velocity. Does this instant center also have zero acceleration? Can one use an instant center approach for determining accelerations of points on a rigid body?

no

ANSWERS:

- Generally speaking, the acceleration for an instant center is not zero. The location of the instant center for a rigid body changes with the orientation of the body. Therefore, we would expect the instant center point to be accelerating. As an example, let's consider the rolling without slipping example from earlier on. The contact point C has zero velocity; however, we saw from before that the acceleration of C is not zero since $a_{Cy} \neq 0$.



- If we now consider the acceleration of another point A on the wheel, we have:

$$\vec{a}_A = \vec{a}_C + \vec{\alpha} \times \vec{r}_{A/C} - \omega^2 \vec{r}_{A/C}$$

  Therefore, the total acceleration of point A is the vector sum of the three acceleration components shown in the figure above right. From this we see that the acceleration of A is not generally perpendicular to $\vec{r}_{A/C}$. As a result, C is not the instantaneous center for acceleration. As a general rule, you cannot use instant centers when determining accelerations.

Consider the following approach for determining the instant center for a body and how to visualize the motion of that body about its instant center.

Method: Instant Centers

Recommended steps for instant center analysis for planar motion of a rigid body:

1. Locate two points A and B on the rigid body for which you know some information of their motion.
   - Let A be a point for which you know BOTH the magnitude and direction of its velocity, $\vec{v}_A$.
   - Let B be a point for which you know the direction of its velocity, $\vec{v}_B$.
2. On a sketch of the body, draw the directions of the velocity vectors $\vec{v}_A$ and $\vec{v}_B$.
3. Draw lines that are perpendicular to the respective directions of $\vec{v}_A$ and $\vec{v}_B$.
4. The intersection of the two lines drawn above locates the instant center C of the body. That is, for the instant corresponding to the position of the body in your sketch, we know that $\vec{v}_C = \vec{0}$.
5. From this, we can find:
   - magnitude of the angular velocity of the body, ω, from:

     $$\omega = \frac{|\vec{v}_A|}{\left|\vec{r}_{A/C}\right|}$$

     where $\left|\vec{r}_{A/C}\right|$ is the distance from C to A. Recall from above that we have assumed that we know the speed of point A.
   - the direction of the angular velocity from examining the figure. Since C is the instant center of the body, the body is instantaneously rotating about C. Knowing the direction of $\vec{v}_A$, we can determine if the body is rotating clockwise or counterclockwise about C.
6. The velocity of ANY point D on the body is perpendicular to the line connecting C and D. The speed of D is found from:

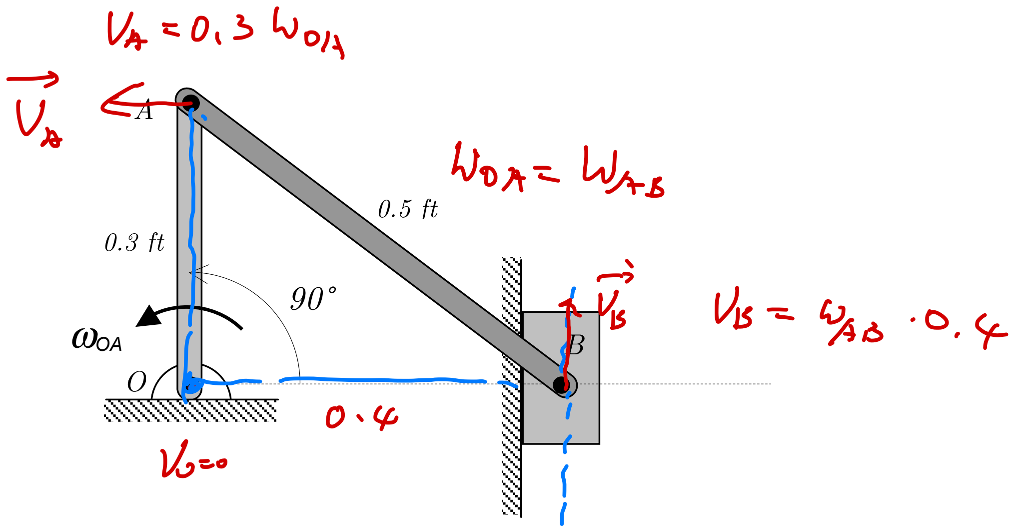
   $$|\vec{v}_D| = |\vec{\omega}| \left|\vec{r}_{D/C}\right|$$

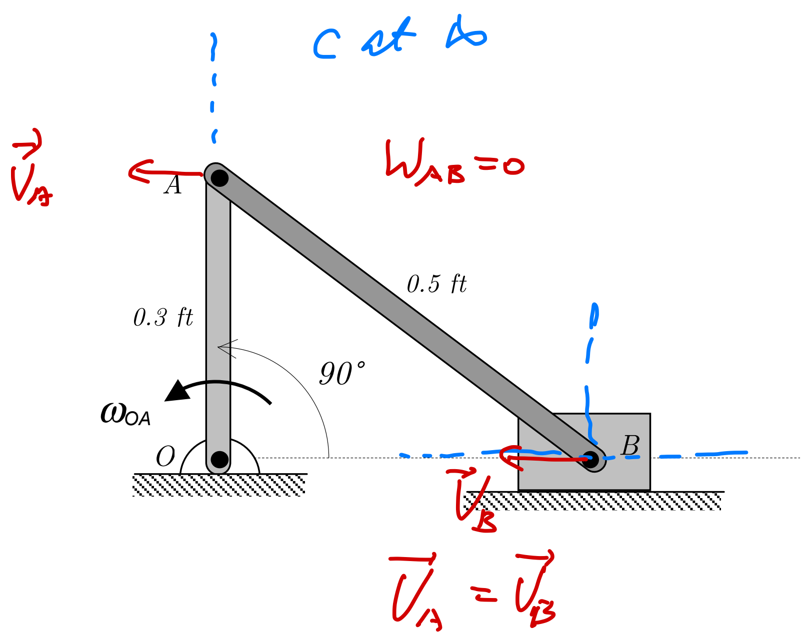
   The direction of is found from recalling again that the body is instantaneously rotating about point C.

Using Instant Center Locations for *Qualitative* Assessment of Motion

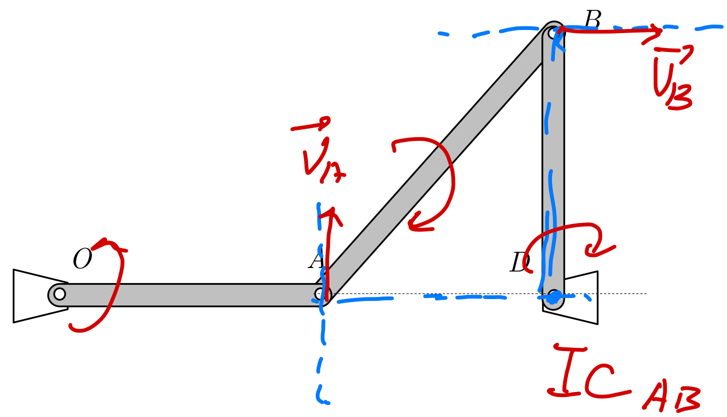
Knowing the location of the instant centers for the links of a mechanism often allows you to understand the qualitative nature of the motion of the mechanism without doing quantitative calculations. Consider the following four instant center examples.

What is the sense of rotation for link AB for the following two mechanisms at the positions shown?

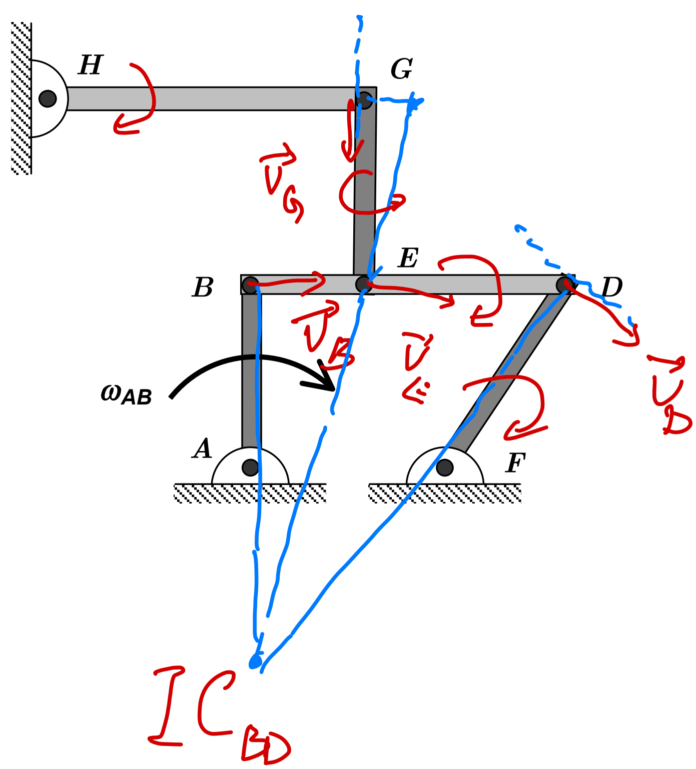




What is the sense of rotation of links OA and BD, if link AB is rotating CW?



Link AB is rotating in the clockwise direction. What is the sense of rotation for the other links of this mechanism?



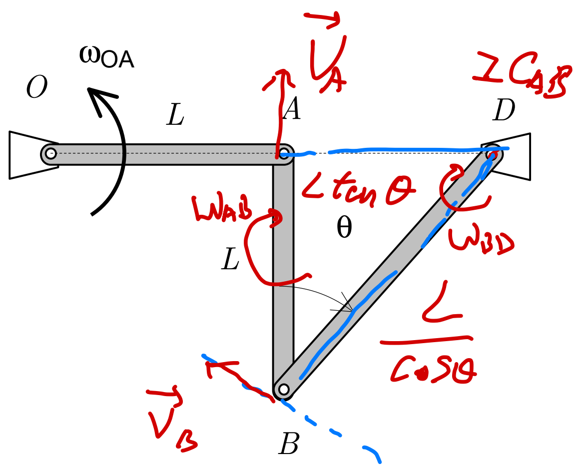
Example 2.B.1

Given: Link OA rotates with an angular speed of $\omega_{OA} = 3$ rad/s with a counterclockwise sense about pin O. At the instant shown, link OA is horizontal, AB is vertical and $\theta = 36.87°$.

Find:

(a) Locate the instant center IC_{AB} for link AB.

(b) Using the location of IC_{AB}, determine the angular velocities of links AB and DB.



$$|\vec{V}_A| = \omega_{OA} \cdot L$$

$$\omega_{AB} = \frac{|\vec{V}_A|}{L\tan\theta}$$

$$|\vec{V}_B| = \omega_{AB} \cdot \frac{L}{\cos\theta}$$

$$\omega_{BD} = \omega_{AB}$$

$$\vec{\omega}_{AB} = \vec{\omega}_{BD} = -\frac{|\vec{V}_A|}{L\tan\theta}\hat{k}$$

Example 2.B.2

Given: A cable, wrapped around the inner radius of the pulley shown, is being raised at a rate of v_A. A second cable is wrapped around the outer radius of the same pulley with the upper end of this cable attached to ground. Assume that the pulley does not slip on either cable.

Find: Determine:

(a) The location of the instant center for the pulley; and at C

(b) The velocity of point B on the outer radius of the pulley when B is directly above the center O of the pulley. Sketch this velocity vector.

Use the following parameters in your analysis: $v_A = 3$ m/s, $r = 0.5$ ft and $R = 1$ ft.

$$\omega = \frac{V_A}{R+r}$$

$$V_B = \omega \cdot \sqrt{2}R = \sqrt{2}\,\frac{V_A}{R+r}\,R$$

$$\vec{V}_B = -\frac{V_A}{R+r}R\,\hat{i} + \frac{V_A}{R+r}R\,\hat{j}$$