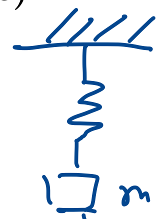
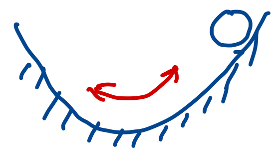
Vibration: periodic oscillatory motion.

$x(t)$ — periodic function

$\rightarrow$ $V(t)$, $a(t)$ — periodic

A. Derivation of Equations of Motion (EOMs)





hold then release

Background

In our earlier work, we have used the Newton-Euler Equations:

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_A = I_A\vec{\alpha}$$

to study the relationship between accelerations and forces acting on systems of particles and rigid bodies. If velocity and position were desired information in a problem, we would then use integration techniques to find these quantities. In almost all situations of interest to us, we solved for acceleration/velocity/position for a given instant in time rather than solving for an expression that was valid for all time.

$\rightarrow$ Look at the whole time history of $x(t)$, $v(t)$, $a(t)$

Objectives

In these lectures we will study the response of systems whose motion is *oscillatory* in nature. For these studies we will focus on understanding the change in position as a general function of time. In particular, we will seek to develop differential equations that govern these oscillatory systems, and use these equations to derive an expression for their oscillatory motion.

General Procedure:

① derive differential Equations (Equation of Motion) based on Newton/Euler (or work, impulse, etc.)

② Solve the differential Equations for

$x(t)$, $v(t)$, $a(t)$ ...

Lecture Material

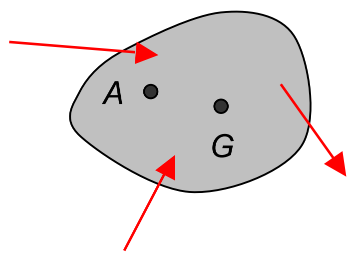
Vibrational systems are typically made up of the following components: particles/rigid bodies (mass/inertia), springs, dashpots and external forcing. We will consider some key points related to these components in the following discussion.

In our previous work we have used the following Newton-Euler Equations for kinetics analysis:

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_A = I_A\vec{\alpha}$$

where A is either a fixed point on the body or the center of mass, G, for the body.



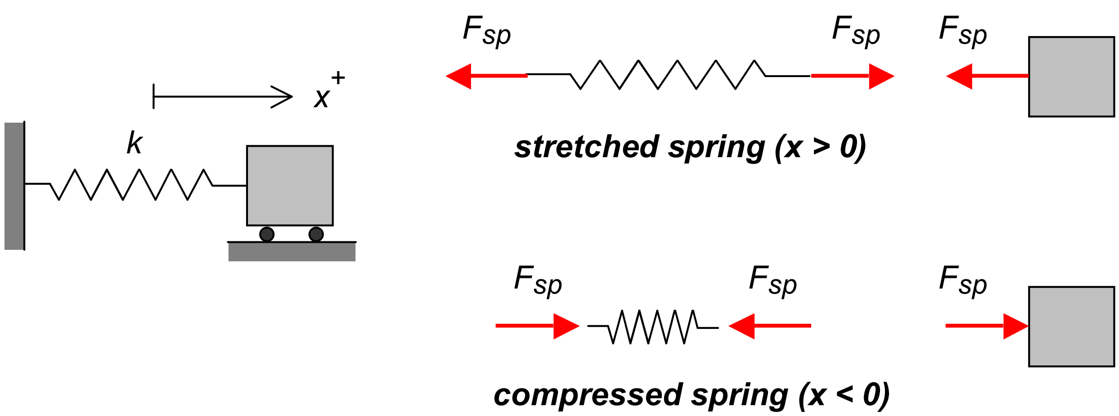
Although we have had a lot of experience this semester in drawing FBDs and using these FBDs with the Newton-Euler Equations, some additional discussion is needed related to two types of forces that are important in the study of vibrations: spring forces and dashpot forces. Consider the discussion of drawing these forces on FBDs. Give particular attention to the DIRECTION of these forces in your FBDs.

Linear springs

Consider the linear spring of stiffness k acting on the particle shown below. Note that if the spring is stretched, the spring *pulls* on the mass to the left. On the other hand, if the spring is compressed, the spring *pushes* on the mass to the right.

Here we will assume that $x = 0$ corresponds to the position of the particle where the spring is unstretched (neither stretched nor compressed). Furthermore, for the sake of discussion, let's assume that for this instant in time we have $x > 0$, which implies that the spring is stretched. Since the spring is stretched, the spring pulls to the left on the particle as shown in the FBD of the particle below. The magnitude of the force of the spring on the particle given by: $\boxed{F_{sp} = kx}$.

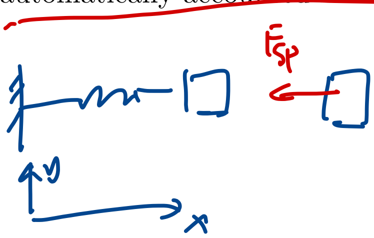
At some other instant in time, the spring might be compressed $(x < 0)$. At this instant, the compressed spring will *push* on the particle, resulting the FBD of the particle shown below with the magnitude of the force of the spring on the particle again given by: $F_{sp} = kx$.

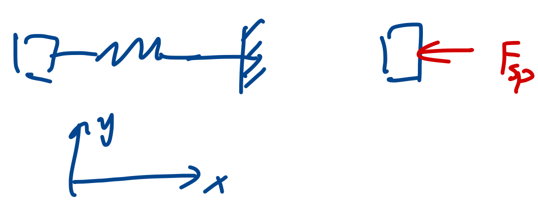


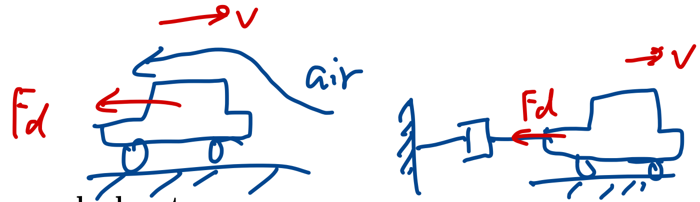
Direction / Sign is very important

During the subsequent motion of a system with springs, the springs will be in tension and compression at different times. However, we need to draw our FBDs in such a way that the FBDs are valid for ANY position of the system. To accomplish this, it is recommended that you use the following procedure when drawing the spring forces:

1. Assume that all coordinates, such as x, to be positive.
2. Based on the positions in step 1, determine whether the spring is in tension or compression.
3. Draw the forces on the bodies in directions that are consistent with the result of tension or compression found in step 2 and label the force as $k\Delta$, where Δ is the amount of tension or compression in the spring.
4. In subsequent positions for which the coordinates are not positive, the signs on the forces will automatically accounted for in the mathematics.







• Fd ↗ as v ↗

• Fd is in opposite direction of v

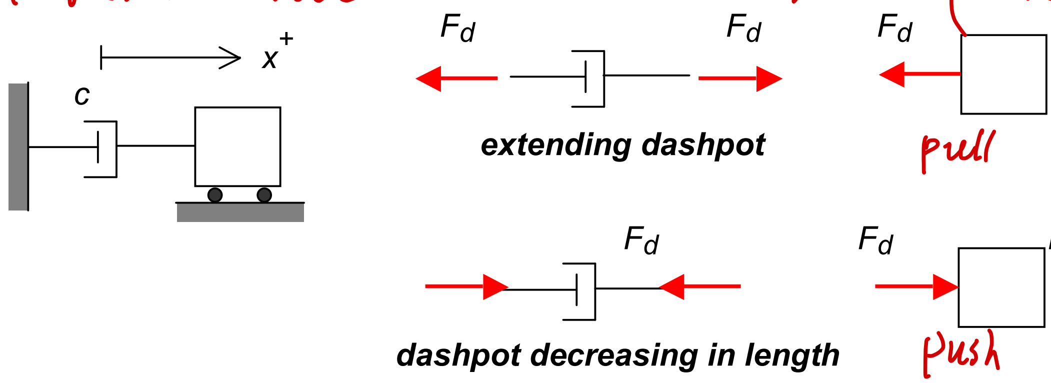
Linear dashpots

Consider the linear dashpot having a damping constant of c which acts on the particle shown below. If the dashpot is extending (increasing in length), the dashpot *pulls* on the mass to the left. On the other hand, if the dashpot is decreasing in length, the dashpot *pushes* on the mass to the right.

Let's assume that for this instant in time we have $\dot{x} > 0$, which implies that the length of the dashpot is extending, and the dashpot pulls to the left on the particle. Therefore, the force of the dashpot on the particle points to the left, as shown in the FBD of the particle below with the magnitude of the force of the dashpot on the particle given by: $F_d = c\dot{x}$.

At some other instant in time, the dashpot might be decreasing in length ($\dot{x} < 0$). At this instant, the dashpot will *push* on the particle, resulting the FBD of the particle shown below with the magnitude of the force of the dashpot on the particle given by: $F_d = c\dot{x}$.

Don't memorize rules but think about physics: (dashpot is a resistance force.)



During the motion of a system with dashpots, the dashpots will be increasing and decreasing in length at different times of the motion. However, we need to draw our FBDs in such a way that they are valid for ANY position of the system. To accomplish this, it is recommended that you use the following procedure when drawing the dashpot forces:

1. Assume that all velocities, such as $\dot{x}$, to be positive.

2. Based on the positions in step 1, determine whether the dashpot is increasing or decreasing in length.

3. Draw the forces on the bodies in directions (push or pull) that are consistent with the result of found in step 2 and label the force as $F_d = c\dot{x}$.

4. In subsequent positions for which the coordinates are not positive, the signs on the forces will automatically accounted for in the mathematics.

Discussion

For of the vibrational systems that we will study in this course, a single coordinate will be needed to describe the motion of the system. In deriving the equation of motion for these systems, more than one coordinate might be used. However, we will need to enforce all of the kinematic constraints that exist among these coordinates and we will need to reduce the dynamical equations to a SINGLE differential equation of motion (EOM) for the system. This process is best seen by working a number of examples, such as those that follow. In all of these cases, the EOMs will take on the general form of:

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

$$\Sigma F = \underbrace{kx}_{\text{spring}} + \underbrace{c\dot{x}}_{\text{damping}} + \underbrace{f(t)}_{\text{external}} = m\ddot{x}$$

The physical interpretation of M, C, K and $f(t)$ will depend on the individual problem.

The process of deriving the EOM for a given problem is summarized by the following steps (these steps a very similar to those used throughout our study of kinetics in this course...):

1. Define a set of coordinates that you will use to describe the motion of the system. Draw an FBD of each body in the system. For spring and dashpot forces, follow the process described in the preceding pages to determine the directions of these forces.

2. Based on your FBDs in step 2, use the Newton-Euler Equations to write down the dynamical equations for these bodies.

3. Write down the needed kinematic equations needed to relate the coordinates used.

4. Use the results of steps 2 and 3 above to reduce the set of equations to a single, differential equation of motion. This EOM will take on the form shown above.

In the following sections of notes, we will seek to solve this differential equation of motion.

Before leaving this section, it is important to note that the signs of the coefficients M, C and K in the above form of the differential equation of motion (EOM) should all be of the same sign (either all positive or all negative). If the coefficients in your EOM do not abide by this rule, you have made an error in deriving the EOM. Should this occur, do not simply change the signs; generally a sign error indicates a more systemic error in your derivation, for which a change in sign will likely not fix the error. We will see later on why these coefficients must be of the same sign.

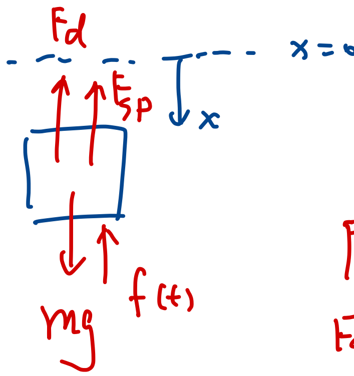
Example 6.A.1

Assume $x=0 \rightarrow$ original length of spring

Given: A particle of mass m is supported by a spring of stiffness k and a dashpot with damping constant c. A vertical force $f(t)$ acts on the particle as shown. Let x describe the position of the particle, where x is measured from the position of the particle when the spring is unstretched.

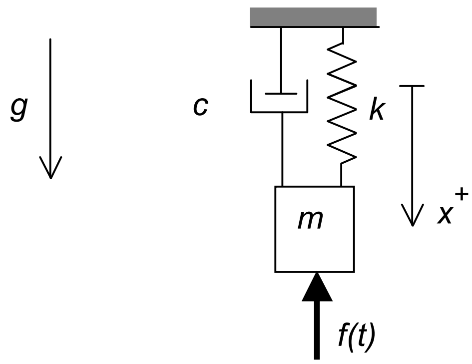
Find: Determine the EOM of this system in terms of the coordinate x.

1 FBD:



$F_d = c\dot{x}$

$F_{sp} = kx$



Newton: $\sum F_x = m\ddot{x}$

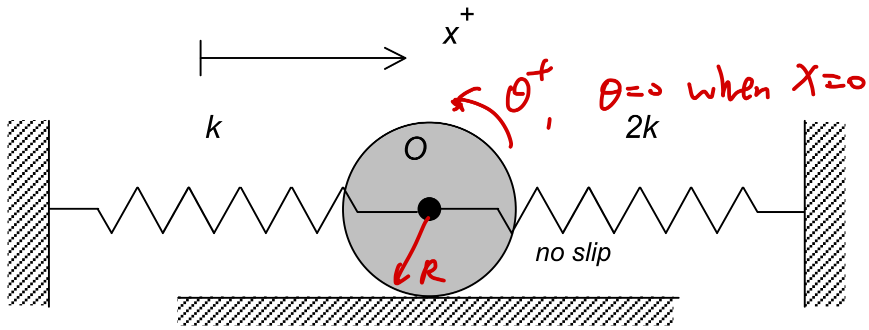
$$mg - f(t) - c\dot{x} - kx = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = mg - f(t) \rightarrow \text{EOM}$$

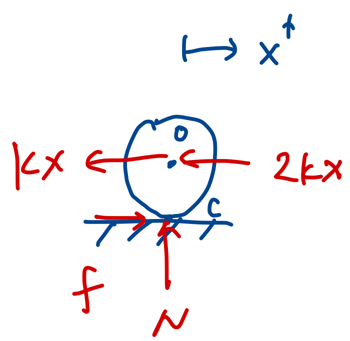
Example 6.A.2

Given: A homogeneous disk of mass m and radius r rolls without slipping on a rough horizontal surface. Two springs, having stiffnesses of k and $2k$, are attached between the disk center O and ground, as shown below. Let x describe the position of O, where the springs are unstretched when $x = 0$.

Find: Determine the EOM for the disk in terms of the coordinate x.



FBD:



$$\Sigma M_C = I_C \ddot{\theta}$$

$\hat{k}$: $kx \cdot R + 2kx \cdot R = I_C \ddot{\theta}$, $I_C = I_O + mR^2$

$= \frac{1}{2}mR^2 + mR^2$

→ 2 unknowns

Kinematics:

$$V_O = \dot{x}\hat{i} = V_C + \omega \times \vec{r}_{O/C}$$

$$\dot{x}\hat{i} = 0 + \dot{\theta}\hat{k} \times (R\hat{j})$$

$$\Rightarrow \dot{x} = -\dot{\theta} R \qquad \Rightarrow 3kRx + I_C \cdot \frac{\ddot{x}}{R} = 0$$