Discussion: Newton-Euler Equations

$$\sum \vec{F} = m\vec{a}_G \qquad \text{(Newton)}$$

$$\sum \vec{M}_A = I_A\vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A) \qquad \text{(Euler)}$$

1. The above are known as the Newton-Euler equations for the dynamic analysis of planar motion of rigid bodies.

2. For the right-hand side of the Newton equation you MUST use the acceleration of the body's center of mass G. There are not exceptions to this rule!

3. In the Euler Equation, you can use ANY point A on the rigid body. YOU are free to choose any point A on the body that you want. Regardless of the choice of point A, you must be consistent in the subscripts:

$$\sum \vec{M}_{\boxed{A}} = I_{\boxed{A}}\vec{\alpha} + \vec{r}_{G/\boxed{A}} \times \left(m\vec{a}_{\boxed{A}}\right)$$

SAME point A for <u>ALL</u> terms

   Specifically:

   - You must use the same point A for the mass moment of inertia I_A as the point that you use for summing the moments $\vec{M}_A$.
   - The acceleration $\vec{a}_A$ must be of the same point A as the point that you use for summing moments $\vec{M}_A$.
   - The vector $\vec{r}_{G/A}$ must extend FROM point A TO the mass center G.

4. Recall there are special forms of Euler's Equation when A is either the center of mass G or a fixed point O on the rigid body. In these special cases the last term on the right-hand side of this equation vanishes. These are the preferred forms of Euler's Equation because of their simplicity.

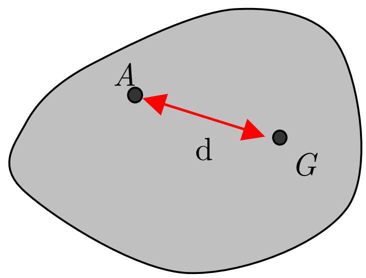
5. IMPORTANT POINT: You must use the same sign conventions on forces as you do for accelerations in the Newton Equation. Similarly, you must use the same sign convention on moments as you do for angular accelerations in the Euler Equation. For the Euler Equation, it is recommended that you use a right-hand rule convention for the moments/angular accelerations.

6. The mass moments of inertia for simple shapes of bodies are listed herein, as well as in many textbooks. Note that mass moments of inertia are generally given for the center of mass G of the body, I_G. In order to find the mass moment of inertia for an arbitrary point A, I_A, you need to use the "parallel axis theorem":

$$I_A = I_G + md_{AG}^2$$

In the statement of the parallel axis theorem above, it is noted that points A and G cannot be interchanged. That is, $I_G \neq I_A + md_{AG}^2$; I_G must appear on the same side of the equation as the $+md_{AG}^2$ term.

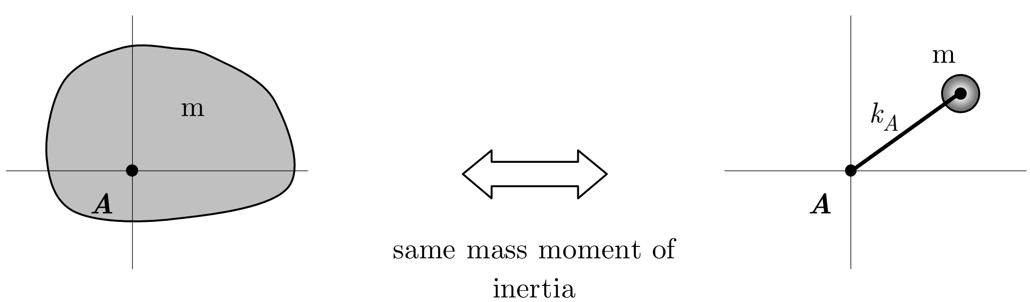
7. From the parallel axis theorem above, we see that $I_A > I_G$ since $I_G > 0$. That is, I_G is the smallest possible mass moment of inertia of the rigid body of all points on the body.



8. The “radius of gyration” of a body of mass m about point A is defined as: $k_A = \sqrt{I_A/m}$. In this course, you will occasionally be given the radius of gyration about some point A on the body and the mass of the body. In order to find the mass moment of inertia about point A, you simply need to re-arrange the above equation to find:

$$I_A = m{k_A}^2$$

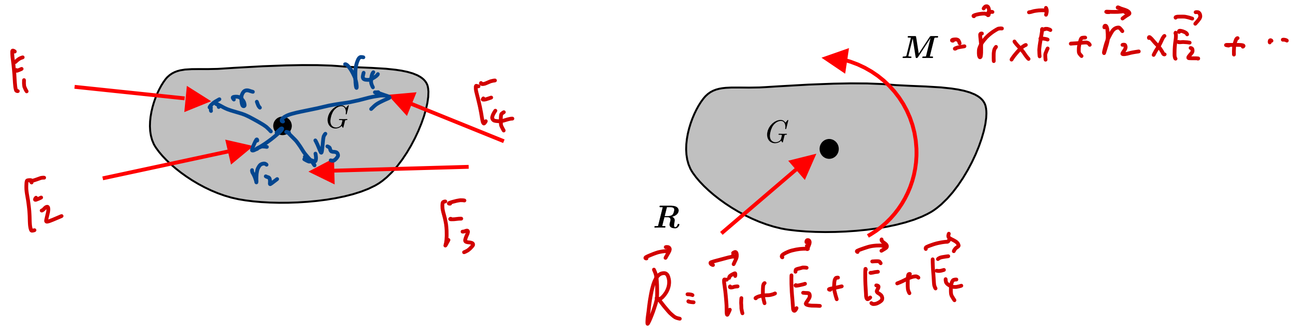
This is all you really need to know about the radius of gyration for this course. However, one can construct a physical interpretation of this quantity. Consider a particle of mass m attached to a massless rod (of length r) pinned to ground at A. As defined above, the mass moment of inertia of this particle about A is given by $I_A = mr^2$. Therefore, the radius of gyration for a rigid body is defined as the distance at which one can concentrate the total mass of the body and have the same mass moment of inertia about A as the original rigid body.



find a radius so that a particle with the same mass will have the same mass moment of inertia as the rigid body.

Interpretation of the Newton-Euler Equations

Recall from statics that forces acting on a rigid body can be moved to a single point so long as you also add a couple that is equal to the summation of moments of those forces about that point (this is called the *equivalent force-couple system*):



- The Newton Equation $\vec{R} = \sum \vec{F} = m\vec{a}_G$ says that we are effectively treating the rigid body as a particle at G in terms of translation motion of G. In other words, the motion of the center of mass G is the same motion as that of a particle of mass m with a force of $\vec{R}$ acting on it.

- The Euler Equation $\vec{M} = \sum \vec{M}_G = I_G\vec{\alpha}$ represents the rotation motion of the body about its center of mass. In other words, the rotational motion of the rigid body is the same rotational motion as if the same rigid body pinned to ground at G has $\vec{M}$ acting on it.

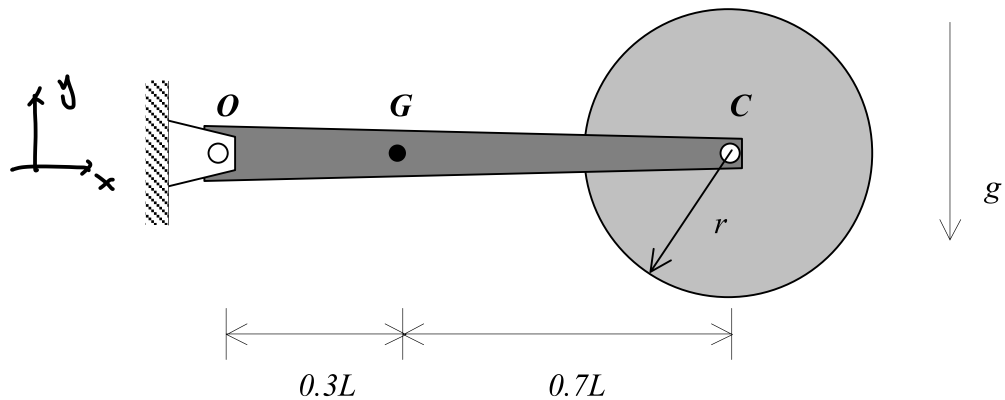
↳ rotation about G is the same.

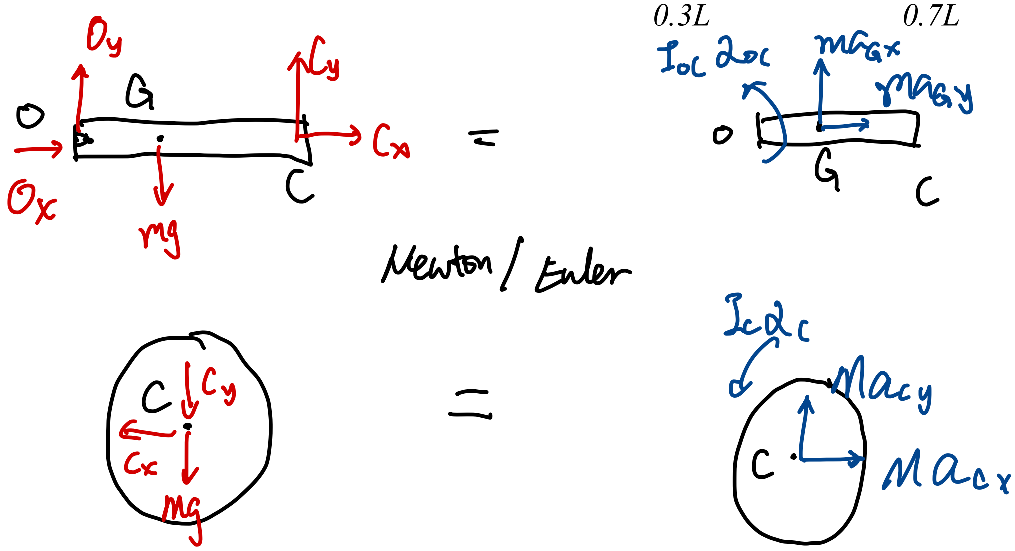
about point G : $I_G = m k_G^2$

Example 5.A.10

Given: An inhomogeneous bar OC (having a length of $L = 2$ m, mass $m = 10$ kg, center of mass at G and centroidal radius of gyration of $k_G = 0.4$ m) is pinned to ground at end O. A homogeneous disk (having an outer radius of $r = 0.6$ m and mass $M = 15$ kg) is pinned to the right end of the bar at the disk's center C. The system is released from rest with OC being horizontal. Assume that all bearings are smooth.

Find: Determine the angular acceleration of arm OC on release.





OC:

$$\Sigma F_x = m a_{Gx} \Rightarrow O_x + C_x = m a_{Gx} \quad (1)$$

$$\Sigma F_y = m a_{Gy} \Rightarrow O_y - mg + C_y = m a_{Gy} \quad (2)$$

$$\Sigma M_G = I_{OC}\alpha_{OC} \Rightarrow -O_y(0.3L) + C_y(0.7L) = I_{OC}\alpha_{OC} = m k_G^2 \alpha_{OC} \quad (3)$$

Disk C:

$$\sum F_x = m a_{cx} \quad \Rightarrow \quad -C_x = m\,\underline{a_{cx}} \qquad (4)$$

$$\sum F_y = m a_{cy} \quad \Rightarrow \quad -C_y - mg = m\,\underline{a_{cy}} \qquad (5)$$

$$\sum M_C = I_C \alpha_C \quad \Rightarrow \quad 0 = I_C \alpha_C \quad \Rightarrow \quad \alpha_C = 0$$

→ 5 Eq.s, 9 unkowns.

Kinematic relations:

release from rest

$$\vec{a}_G = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha}_{OC} \times \vec{r}_{G/O} - \cancel{\omega^2}^{\,0}\, \vec{r}_{G/O}$$

$$\Rightarrow \quad a_{Gx}\hat{i} + a_{Gy}\hat{j} = \alpha_{OC}\hat{k} \times (0.3L\,\hat{i}) = 0.3L\,\alpha_{OC}\,\hat{j}$$

$$\Rightarrow \quad \begin{cases} a_{Gx} = 0 & (6) \\ a_{Gy} = 0.3L\,\alpha_{OC} & (7) \end{cases}$$

$$\vec{a}_C = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha}_{OC} \times \vec{r}_{C/O} - \cancel{\omega^2}^{\,0}\, \vec{r}_{C/O}$$

$$\Rightarrow \quad a_{cx}\hat{i} + a_{cy}\hat{j} = \alpha_{OC}\hat{k} \times (L\,\hat{i}) = L\,\alpha_{OC}\,\hat{j}$$

$$\Rightarrow \quad \begin{cases} a_{cx} = 0 & (8) \\ a_{cy} = L\,\alpha_{OC} & (9) \end{cases}$$

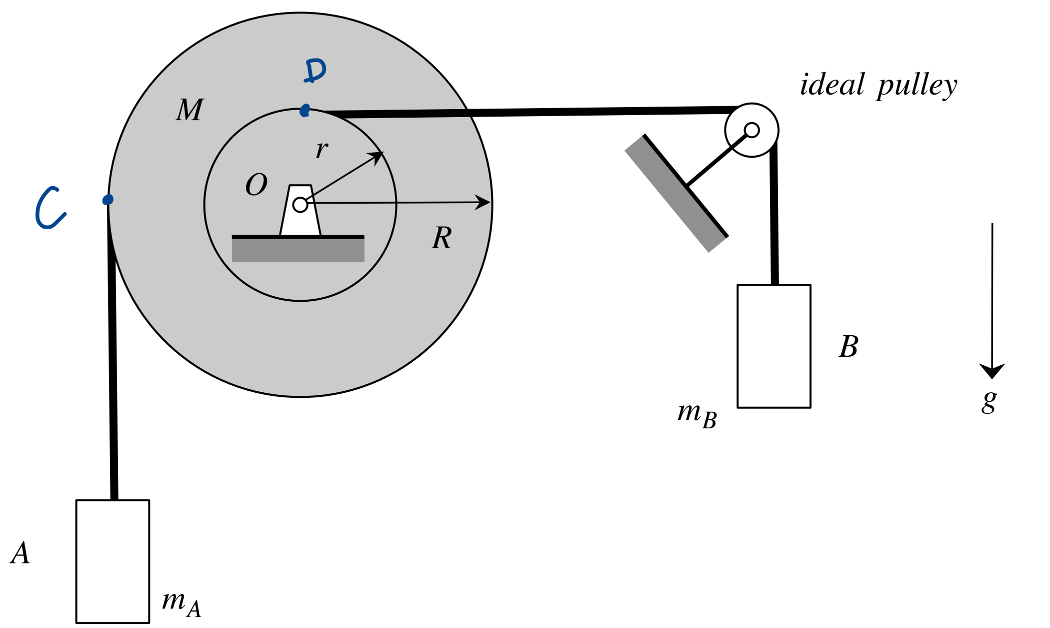
$\Rightarrow$ Solve for α_{OC}

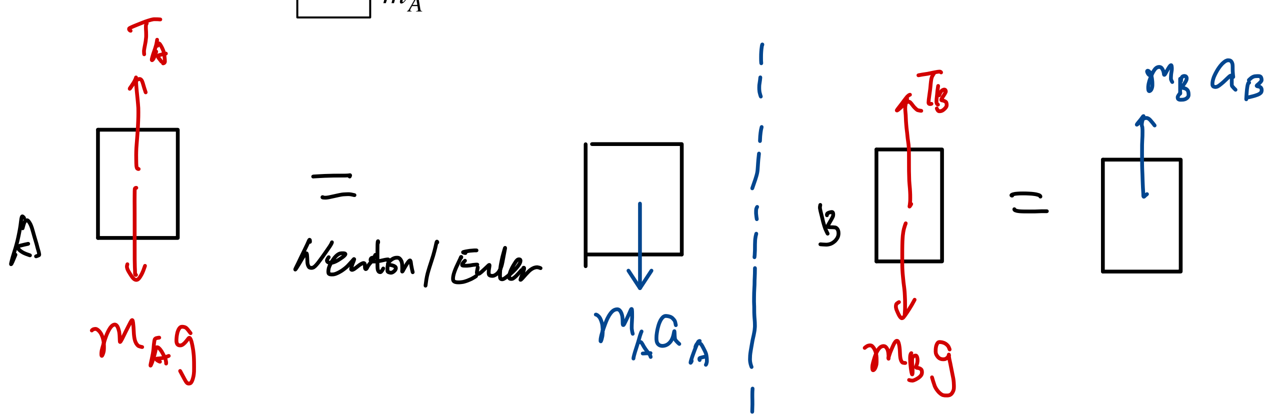
Example 5.A.11

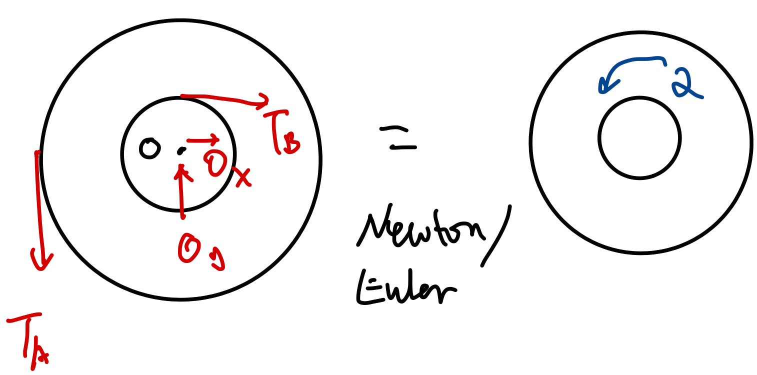
Given: A stepped drum (having a mass of M and radius of gyration about its center O of k_O) is attached to a smooth shaft passing through its center O. A cable wrapped around the outer radius of the drum is attached to block A. A second cable is wrapped around the inner radius of the drum with this cable pulled over an ideal pulley and is attached to block B. Assume that the cables do not slip on the drum. The system is released from rest.

Find: Determine the angular acceleration of the drum on release. Write your answer as a vector.

Use the following parameters in your analysis: m_A = 10 kg, m_B = 30 kg, M = 20 kg, r = 0.2 m, R = 0.4 m and k_O = 0.25 m.







$Ma_{Ox} = 0$

$Ma_{Oy} = 0$

Disk: $\sum M_O = I_O \alpha \Rightarrow T_A R - T_B \cdot r = I_O \alpha$ ①

$\sum F_x = M a_{Ox} = 0 \Rightarrow T_B + O_x = 0$

$\sum F_y = M a_{Oy} = 0 \Rightarrow -T_A + O_y = 0$

$\Big\} \rightarrow$ NOT interesting

A: $\sum F_y = -m_A a_A \Rightarrow T_A - m_A g = -m_A a_A$ ②

B: $\sum F_y = m_B a_B \Rightarrow T_B - m_B g = m_B a_B$ ③

→ 3 eq.s , 5 unknowns

No slipping $\Rightarrow a_{Cy} = -a_A \quad , \quad a_{Dx} = -a_B$

Rigid Body :

$$\vec{a}_C = \cancelto{0}{\vec{a}_O} + \vec{\alpha} \times \vec{r}_{C/O} - \cancelto{0}{\omega^2} \vec{r}_{C/O}$$

$$\Rightarrow a_{Cx}\hat{i} + a_{Cy}\hat{j} = \alpha \hat{k} \times (-R\hat{i}) = -\alpha R \hat{j}$$

$$\Rightarrow a_A = -a_{Cy} = \alpha R \quad ④$$

$$\vec{a}_D = \vec{a}_O + \vec{\alpha} \times \vec{r}_{D/O} - \omega^2 \vec{r}_{D/O}$$

$$\Rightarrow a_{Dx}\hat{i} + a_{Dy}\hat{j} = -\alpha r \hat{i}$$

$$\Rightarrow a_B = -a_{Dx} = \alpha r \quad ⑤$$

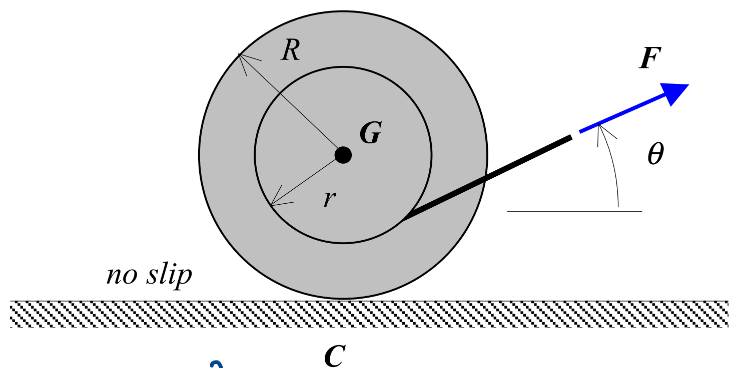
→ Solve for α

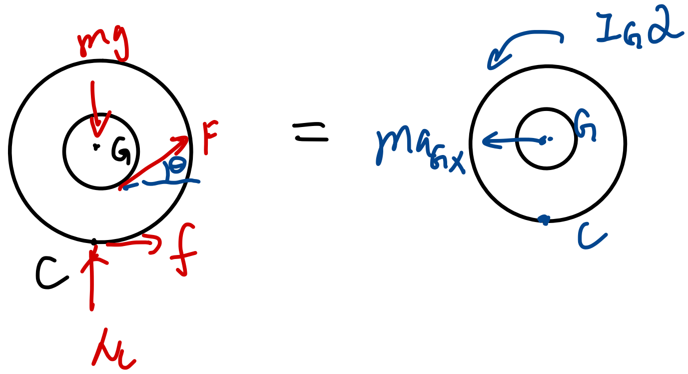
Example 5.A.12

Given: A spool has a mass of $m = 30$ kg, a centroidal radius of gyration $k_G = 0.25$ m, an outer radius of $R = 0.3$ m and an inner radius of $r = 0.1$ m. A constant force $F = 60$ N is applied at an angle of θ by a cord that is wrapped around the inner radius of the spool. The spool rolls without slipping on the rough horizontal surface.

$\omega \neq 0$

Find: Determine the angular acceleration of the spool as a function of the angle θ.





$a_{Gy} = 0$

No slipping $\Rightarrow \begin{cases} a_{Cx} = 0 \\ a_{Cy} \neq 0 \end{cases}$

$\Sigma F_x = m a_{Gx} \Rightarrow F\cos\theta + f = m\, a_{Gx}$ ①

$\Sigma F_y = m a_{Gy} = 0 \Rightarrow N_c - mg + F\sin\theta = 0$ ②

$\Sigma M_G = I_G \alpha \Rightarrow F \cdot r + f R = I_G \alpha$ ③

kinematics relation:

$$\vec{a}_G = \vec{a}_c + \vec{\alpha} \times \vec{r}_{G/C} - \omega^2 \vec{r}_{G/C}$$

$$\Rightarrow a_{Gx}\hat{i} = a_{Cy}\hat{j} - \alpha R\hat{i} - \omega^2 R\hat{j}$$

$\hat{i}$: $a_{Gx} = -\alpha R$ ④ $\rightarrow$ Solve for α