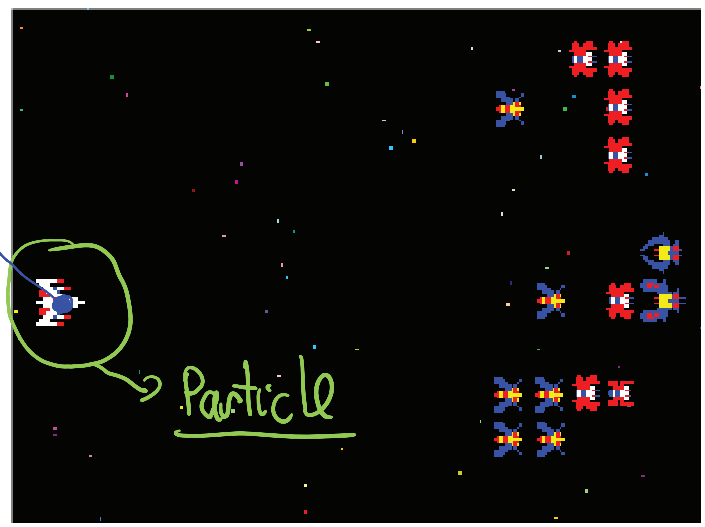
Chapter 5

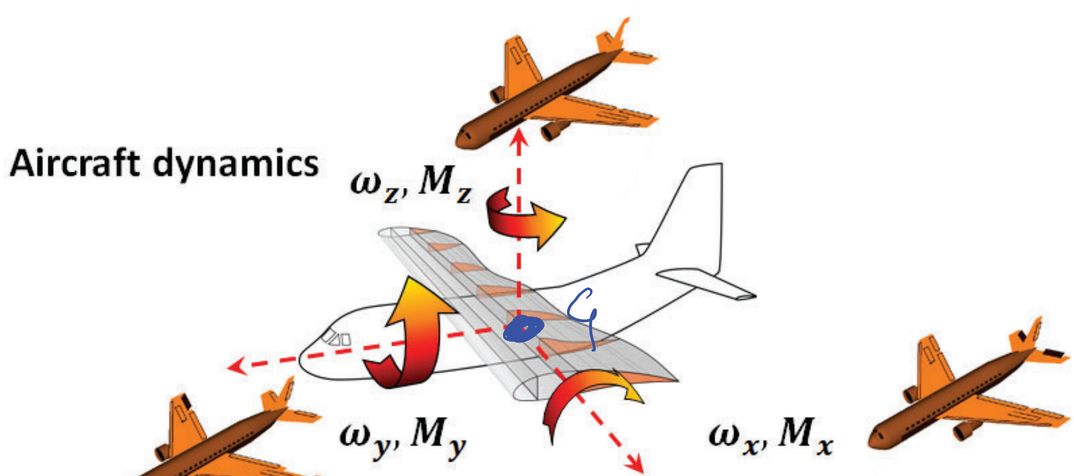
Planar Rigid Body Kinetics

- Lecture 26 Objectives:
  - Introduce Kinetics of rigid bodies
  - Analysis using Newton-Euler equations for kinetics of rigid bodies

$\sum \vec{F} = m\vec{a}$



Particle



https://en.wikipedia.org/wiki/Flight_dynamics_(fixed-wing_aircraft)

rigid body

$\sum \vec{F} = m\vec{a}$

$\sum \vec{M} =$

A. Rigid Body Kinetics: The Newton-Euler Equations

Background

In our earlier studies of the kinetics of particles, we have used the following set of equations for a single particle i (Newton's Second Law and the angular momentum equation):

$$\vec{F}_i = m_i \vec{a}_i$$

$$\vec{M}_{Oi} = \frac{d}{dt}\left[\vec{r}_{i/O} \times (m_i \vec{v}_i)\right] \qquad ; \quad \text{O is a FIXED point}$$

} Particles

We also saw that for a SYSTEM of particles, the above equations become:

$$\left(\sum \vec{F}\right)_{ext} = m\vec{a}_G$$

$$\left(\sum \vec{M}_A\right)_{ext} = \frac{d}{dt}\sum_i \left[\vec{r}_{i/A} \times \left(m_i \vec{v}_{i/A}\right)\right] + \vec{r}_{G/A} \times (m\vec{v}_A)$$

} Rigid Bodies

$d\vec{r}_{B/A}/dt = 0$

where G is the center of mass for the system, A is an arbitrary point in the system, $m = \sum_i m_i$ is the total mass of the system, and $\left(\sum \vec{F}\right)_{ext}$ and $\left(\sum \vec{M}_A\right)_{ext}$ are the total external forces and moments (about point A), respectively, acting on the system.

A continuous body can be thought of as a collection of an infinite set of particles with each particle having an infinitesimal mass. For a continuous body, we can replace the summations over the masses by an integral over the mass of the body:

$$\sum_i (\bullet)\ m_i \rightarrow \int_{vol} (\bullet)\ dm$$

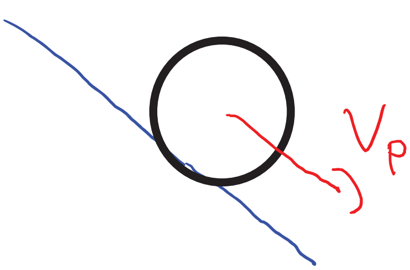
→ Collection of Particles

Objectives

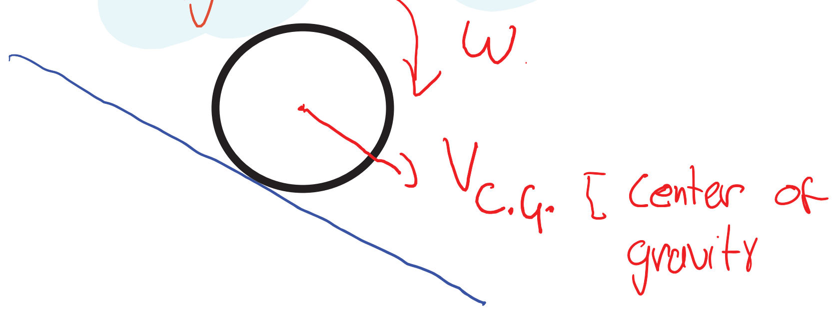
In these lectures our goal is develop and use the set of Newton-Euler equations in solving kinetics problems dealing with planar motion of rigid bodies.

Chapter 5: Rigid Body Kinetics

Particle Analysis

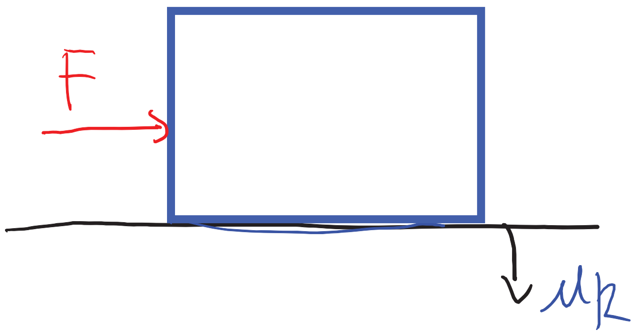


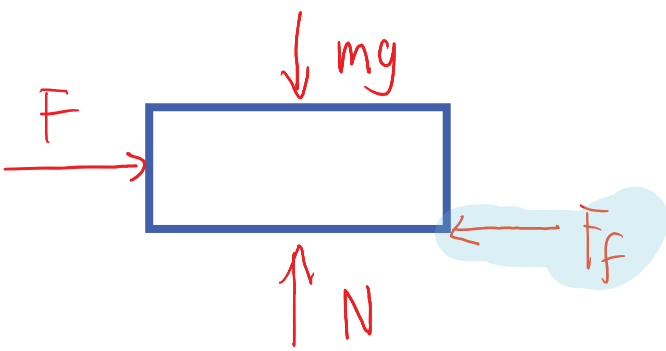
Rigid body Analysis



Chapter 5

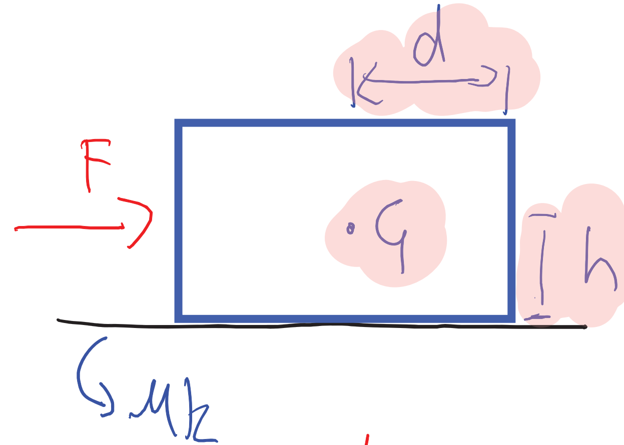
Particle Analysis

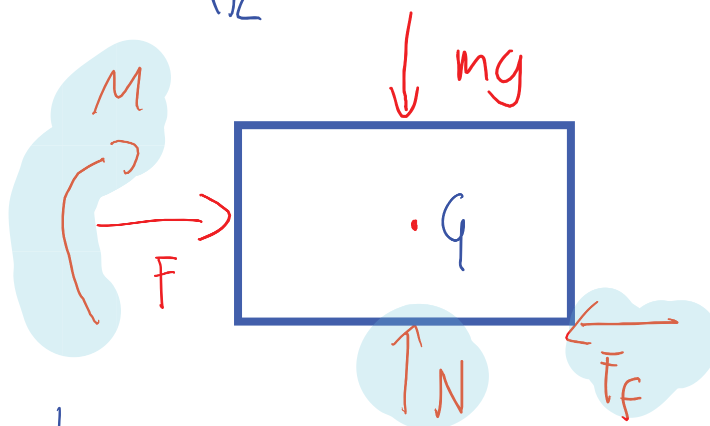




- Point particles
  - no dimensions
  - no rotations

Rigid body Analysis





- Bodies have dimensions
- Forces create moments & rotations

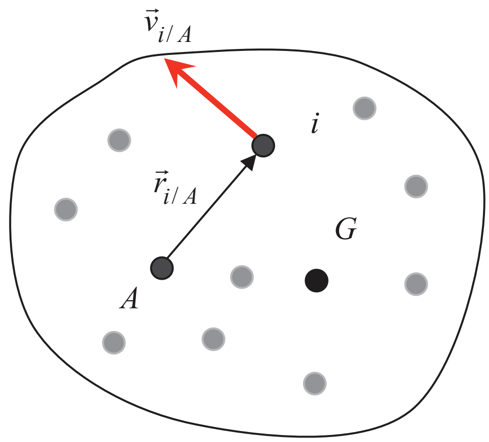
Lecture Material

From the equations in the Background section above, we have the following dynamical equations for a system of particles:

$$\left(\sum \vec{F}\right)_{ext} = m\vec{a}_G$$ → Newton's 2nd Law

$$\left(\sum \vec{M}_A\right)_{ext} = \frac{d}{dt}\sum_i \left[\vec{r}_{i/A} \times \left(m_i \vec{v}_{i/A}\right)\right] + \vec{r}_{G/A} \times \left(m\vec{v}_A\right)$$ → Angular-Impulse Momentum

Ensemble of Particles



system of N particles

To produce an equivalent set of equations for a rigid body, we need to:

- enforce a rigid connection between all points in the system. For this we will use the rigid body velocity equation between the velocity of A and particle i:

  $$\vec{v}_{i/A} = \vec{v}_i - \vec{v}_A = \vec{\omega} \times \vec{r}_{i/A}$$

- envision a rigid body as an infinite set of particles of infinitesimal size for which:

  $$\sum_i (\,\bullet\,)\ m_i \ \rightarrow \int_{vol} (\,\bullet\,)\ dm$$

To this end, we substitute the rigid body velocity kinematics equation into the first term on the right-hand side of the angular momentum equation producing:

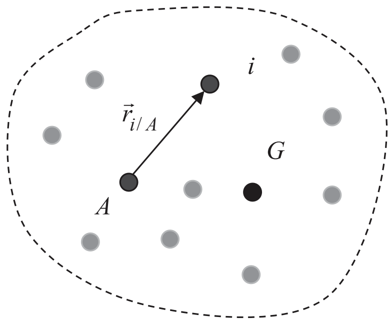
$$\vec{r}_{i/A} \times \left(m_i \vec{v}_{i/A}\right) = m_i \vec{r}_{i/A} \times \left(\vec{\omega} \times \vec{r}_{i/A}\right)$$

From the figures on the following page, we see that the right hand side of the above equation becomes:

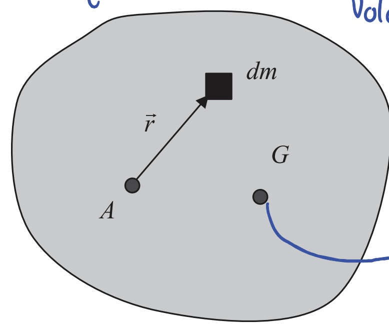
$$\vec{r}_{i/A} \times \left(m_i \vec{v}_{i/A}\right) = m_i \left|\vec{r}_{i/A}\right|^2 \vec{\omega}$$

Rigid body equations require: • $r_{i/A} \rightarrow$ constant

$\longrightarrow$ • $\sum_i (\cdot) m_i \longrightarrow \int_{\text{Volume}} (\cdot)\, dm$



system of rigidly – connected particles



$\rightarrow$ Center of mass/gravity

continuous rigid body

Newton-Euler Equations: $\sum \vec{F} = m\vec{a}_G$ $\longrightarrow$ Newton

New Equation $\longrightarrow \sum \vec{M}_A = I_A \vec{\alpha} + \vec{r}_{G/A} \times m\vec{a}_A$ $\longrightarrow$ Euler Equation

Some Special Forms of the Euler Equation:

• where to take moments about

The moment equation derived above for a rigid body:

$$\left(\sum \vec{M}_A\right)_{ext} = I_A \vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A)$$

is known as "Euler's Equation". Consider the following observations related to the choice of point A for Euler's Equation:

- If you choose A to be the center of mass, G, then $\vec{r}_{G/A} = \vec{0}$. For this case, the above equation reduces to:

$$\sum \vec{M}_G = I_G \vec{\alpha}$$

$\rightarrow$ about center of mass

- If you choose A to be a fixed point on the body, O, then $\vec{a}_O = \vec{0}$. For this case, the above equation reduces to:

$$\sum \vec{M}_O = I_O \vec{\alpha}$$

$\rightarrow$ about fixed Point

- If you choose a point A which has an acceleration vector that is parallel to $\vec{r}_{G/A}$ (and therefore, $\vec{r}_{G/A} \times \vec{a}_A = \vec{0}$), then the above equation reduces to:

$$\sum \vec{M}_A = I_A \vec{\alpha}$$

$\rightarrow$ least common simplifying Point

$\downarrow$

rolling without slip

For all other points, we need to use the full form of the Euler Equation:

$$\sum \vec{M}_A = I_A \vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A)$$

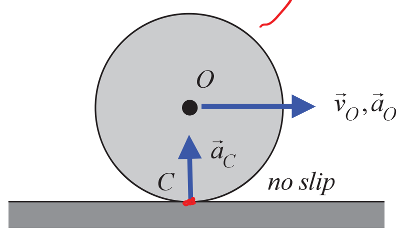
$\rightarrow$ $\vec{r}_{G/A}$ and $\vec{a}_A$ are aligned

CHALLENGE QUESTION: The preceding discussion points out the importance of wisely choosing point A to enable the use of the short version of Euler's Equation:

$$\sum \vec{M}_A = I_A \vec{\alpha}$$

Consider a wheel rolling without slipping on a stationary surface, as shown below. Can you use a no-slip contact point C for this moment equation?

Disk rolling without sliping



- $\vec{a}_c = a\hat{j}$
- $\vec{r}_{G/C} = R\hat{j}$

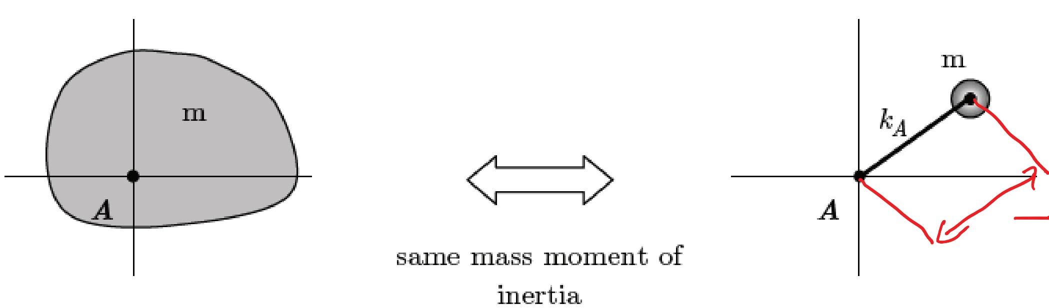
ANSWER: Recall that the acceleration of the no-slip point C has an acceleration that points directly toward the center of the wheel. If the wheel is homogeneous (O is the center of mass G), then $\vec{r}_{G/C}$ is parallel to $\vec{a}_C$, and as a result, we can use $\sum \vec{M}_C = I_C \vec{\alpha}$ for our Euler equation. On the other hand, if the wheel is inhomogeneous with O not being the center of mass, then point C cannot be used. Also, if the wheel slips at C, then the acceleration of C no longer points toward O. For this case, point C cannot be used in the short version of Euler's Equation.

Radius of Gyration:

8. The "radius of gyration" of a body of mass m about point A is defined as: $k_A = \sqrt{I_A/m}$. In this course, you will occasionally be given the radius of gyration about some point A on the body and the mass of the body. In order to find the mass moment of inertia about point A, you simply need to re-arrange the above equation to find:

$$I_A = m{k_A}^2$$

This is all you really need to know about the radius of gyration for this course. However, one can construct a physical interpretation of this quantity. Consider a particle of mass m attached to a massless rod (of length r) pinned to ground at A. As defined above, the mass moment of inertia of this particle about A is given by $I_A = mr^2$. Therefore, the radius of gyration for a rigid body is defined as the distance at which one can concentrate the total mass of the body and have the same mass moment of inertia about A as the original rigid body.



→ radius of Gyration

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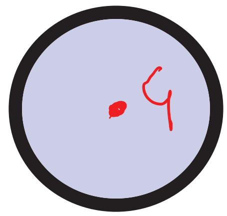
Question C5.9

An inhomogeneous disk has its center of mass at G and is rotating about the geometric center of the disk O. At position I, G is directly above O; at position II, G is directly below O. Let $(F_O)_I$ and $(F_O)_{II}$ represent the magnitudes of the reaction force on the disk at O at positions I and II, respectively. Circle the correct answer below.

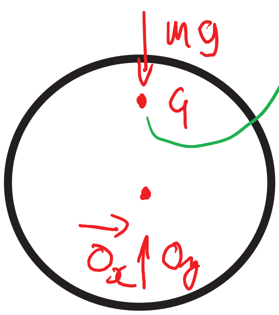
(a) $(F_O)_I > (F_O)_{II}$

(b) $(F_O)_I = (F_O)_{II}$

(c) $(F_O)_I < (F_O)_{II}$

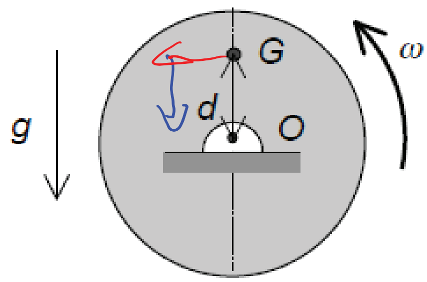


→ Geometric
→ Gravity
Positioned in same Place



Center of gravity

Position I

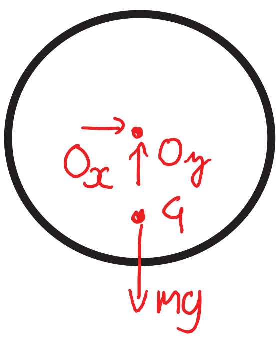


$\sum F_x^I = 0$

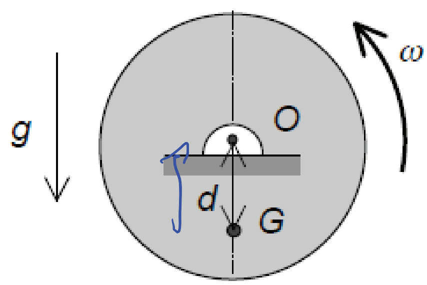
$\sum F_y^I: -mg + O_y = ma_{G_y}$

$\sum M_O = 0$

$O_y = F_O$



Position II



$\sum F_x^{II} = 0$

$\sum F_y^{II}: -mg + O_y = ma_{G_y}$

$\sum M_O^{II} = 0$

$\vec{a}_G = \vec{a}_O + \alpha \times \vec{r}_{G/O} - \omega^2 \vec{r}_{G/O}$ → Ch 2

I: $\vec{a}_G^{I} = -\omega^2 d \, \hat{j}$

II: $\vec{a}_G^{II} = \omega^2 d \, \hat{j}$ → $O_y^{II} = m\omega^2 d + mg$

Mass Moment of Inertia for a Rigid Body

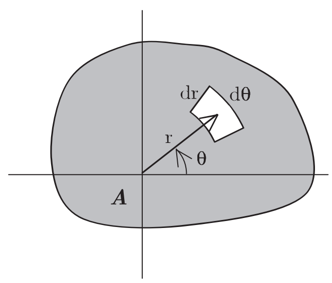
The mass moment of inertia about an arbitrary point A on a rigid body, I_A, has been defined here as:

$$I_A = \int r^2 dm = \int \rho r^2 dV$$

where ρ is the mass density (mass per unit volume) of the body and dV is a differential volume element. The form of this integral will, of course, depend on the coordinate system used. Consider the following forms of this integral for a planar rigid body. In each case, the z-axis is parallel to the axis of rotation for the body and the radial distance r is measured from the origin at point A.

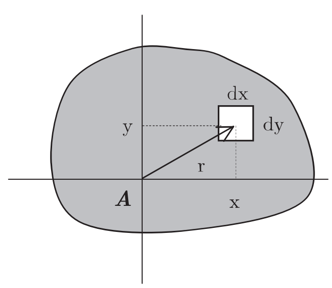
- ***Polar coordinates*** (body having a thickness h): Here we have $dV = hrdrd\theta$. Therefore,

$$I_A = \int\int \rho h r^3 dr d\theta$$



- ***Cartesian coordinates*** (body having a thickness h): Here we have $dV = hdxdy$ and $r = \sqrt{x^2 + y^2}$. Therefore,

$$I_A = \int\int \rho r^2 h dx dy = \int\int \rho h \left(x^2 + y^2\right) dx dy$$

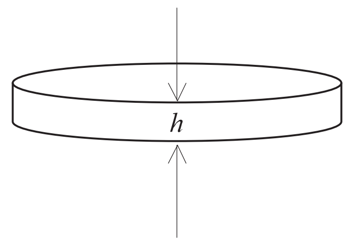


Derivations of Mass Moment of Inertia for Two Commonly Used Objects

For a body of mass density ρ:

$$I_A = \int r^2 dm \qquad ; \qquad dm = \rho \, dV$$

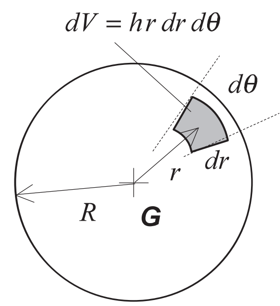
Homogeneous circular disk of mass m and radius R



$$I_G = \int r^2 dm$$

$$= \rho h \int_0^{2\pi} \left(\int_0^r r^3 dr \right) d\theta$$

$$= \rho h \int_0^{2\pi} \left(\frac{1}{4} r^4 \right) d\theta = \rho h \left(\frac{\pi}{2} r^4 \right)$$



Note that the mass of the disk is given by:

$$m = \rho V = \rho \left(\pi r^2 h \right)$$

Therefore, $I_G = \dfrac{1}{2} m r^2$

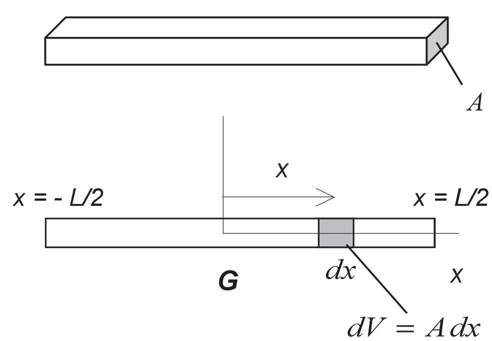
Homogeneous thin bar of mass m and length L

$$I_G = \int r^2 dm = \rho A \int_{-L/2}^{L/2} x^2 dx$$

$$= \rho A \left(\frac{1}{3} r^3 \right)_{x=-L/2}^{x=L/2}$$

$$= \frac{\rho A}{3} \left[\frac{L^3}{8} - \left(-\frac{L^3}{8} \right) \right] = \frac{\rho A L^3}{12}$$

$$m = \rho A L$$



Therefore, $I_G = \dfrac{1}{12} m L^2$

Question C5.2

A block with center of mass at G slides to the left on a rough horizontal surface. Circle the answer below that most accurately describes the location of the normal contact force on the block from the ground as the block slides.

(a) The normal force acts at a point to the left of G

(b) The normal force acts at a point to the right of G

(c) The normal force acts at a point directly beneath G

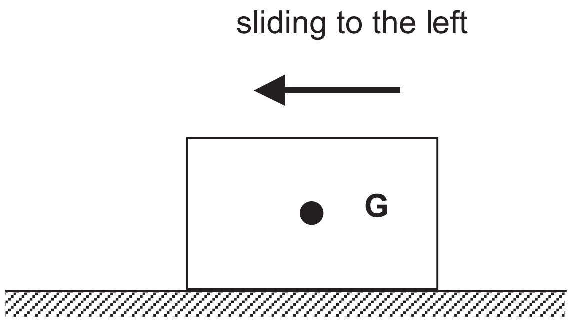
(d) More information is needed to answer this question

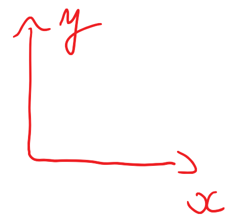
Given :

- rough Surface

Find :

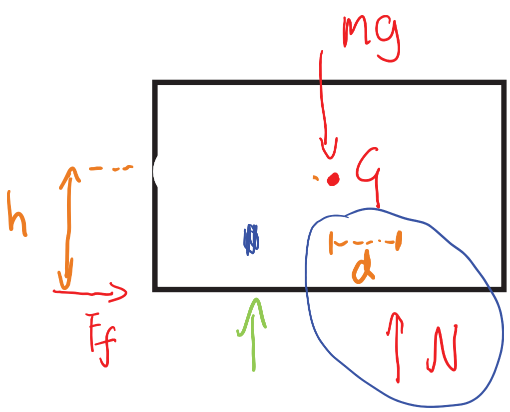
- location Normal force





$F_f = \mu_k N$

1) FBD



- Distance h is certain
- Determine d

2) Kinetics

$\sum F_{y_G}: -mg + N = 0 \rightarrow mg = N \rightarrow$ N direction is up

$\sum F_{x_G}: -F_f = m a_{G_x}$

$\sum M_{[G]}: F_f \cdot h + Nd = I_G \alpha \longrightarrow$ Is the block rotating? $\rightarrow$ α is zero for no rotation

$\hookrightarrow \alpha = 0$

$F_f \cdot h = -N \cdot d$

Conclusion : d must be left of center of gravity