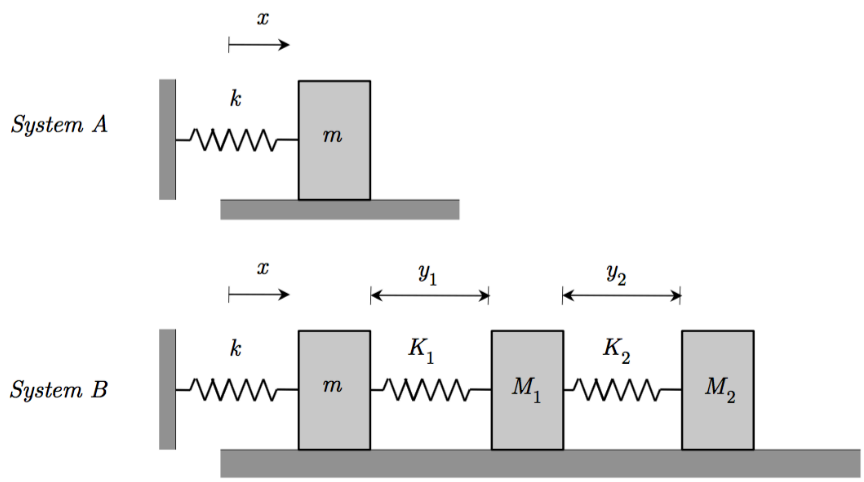
Example V.4.2 – solution



SOLUTION
For System B:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M_1(\dot{x} + \dot{y}_1)^2 + \frac{1}{2}M_2(\dot{x} + \dot{y}_1 + \dot{y}_2)^2$$

$$U = \frac{1}{2}kx^2 + \frac{1}{2}K_1 y_1^2 + \frac{1}{2}K_2 y_2^2$$

EOMs for System B:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0}$$

where $\vec{x} = (x, y_1, y_2)^T$ and

$$[M] = \begin{bmatrix} m + M_1 + M_2 & M_1 + M_2 & M_2 \\ M_1 + M_2 & M_1 + M_2 & M_2 \\ M_2 & M_2 & M_2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_2 \end{bmatrix}$$

The natural frequency for System A is $\omega_A = \sqrt{k/m}$. For System B, we know that the square root of the Rayleigh quotient, $\sqrt{Q(\vec{v})}$, will give an *upper bound* on the exact first natural frequency ω_{1B} for ANY non-trivial trial vector $\vec{v}$. From this, we see that if we can choose a trial vector for which $\sqrt{Q(\vec{v})} < \omega_A$, then we know that the exact value of ω_{1B} will be less than ω_A.

Suppose that we choose:

$$\vec{v} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Using this trial vector:

$$Q(\vec{v}) = \frac{\vec{v}^T[K]\vec{v}}{\vec{v}^T[M]\vec{v}}$$

$$= \frac{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} k & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} m+M_1+M_2 & M_1+M_2 & M_2 \\ M_1+M_2 & M_1+M_2 & M_2 \\ M_2 & M_2 & M_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}$$

$$= \frac{k}{m+M_1+M_2}$$

Therefore:

$$\omega_{1B} \leq \sqrt{Q(\vec{v})} = \sqrt{\frac{k}{m+M_1+M_2}} < \sqrt{\frac{k}{m}} = \omega_A$$

In conclusion, the lowest natural frequency of System B is always less than the natural frequency of System A.