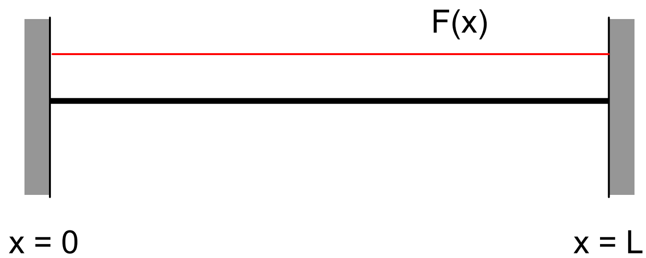
Example IV.5.1

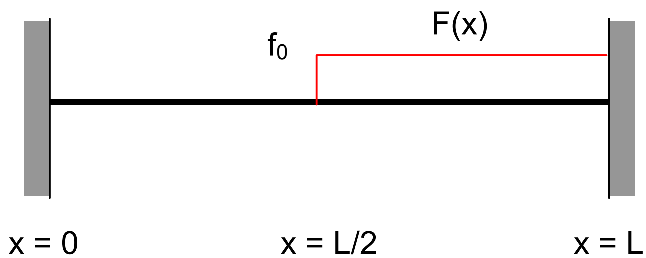
A string that is tautly stretched between two rigid supports is acted upon by a harmonically time varying force, $f(x,t) = f_0\delta(x-a)sin\Omega t$, where $\delta(x-a)$ is the Dirac delta function centered at $x = a$. Find the unoupled modal EOM's for the string.

A string that is tautly stretched between two rigid supports is acted upon by four different types of forcing. Using modal uncoupling, find the forced portion of the response of the string for:

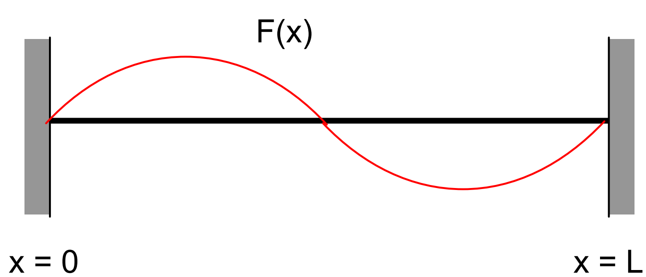
a) $f(x,t) = F(x)\sin\Omega t$ with $F(x) = f_0 = \text{constant}$.



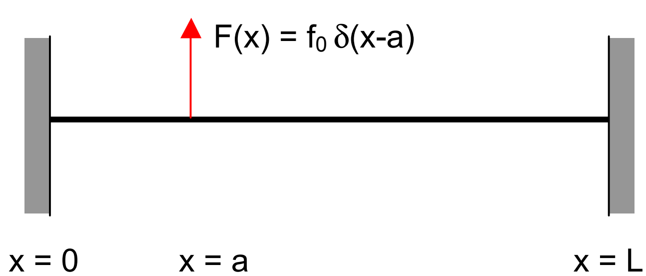
b) $f(x,t) = F(x)\sin\Omega t$ with F(x) shown below.



c) $f(x,t) = F(x)\sin\Omega t$ with $F(x) = f_0 \sin 2\pi \dfrac{x}{L}$



d) $f(x,t) = F(x)\sin\Omega t$ with $F(x) = f_0\delta(x-a)$ where $\delta(x-a)$ is the Dirac delta function[1] centered at x = a.



Modal EOM's:

$$\ddot{q}_j + \omega_j^2 q_j = \hat{f}_j(t)$$

where the modal forcing terms are given by:

$$\begin{aligned}\hat{f}_j(t) &= \int_0^L \tilde{\phi}^{(j)}(x) f(x,t)\, dx \\ &= \left\{ \int_0^L \tilde{\phi}^{(j)}(x) F(x)\, dx \right\} \sin\Omega t \\ &= \hat{F}_j \sin\Omega t\end{aligned}$$

The particular solution of the modal equations is given by:

$$q_{jp}(t) = \frac{\hat{F}_j / \omega_j^2}{1 - \Omega^2/\omega_j^2} \sin\Omega t$$

Therefore,

$$\begin{aligned}u_p(x,t) &= \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x)\, q_{jp}(t) \\ &= \left\{ \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x)\, \frac{\hat{F}_j/\omega_j^2}{1-\Omega^2/\omega_j^2} \right\} \sin\Omega t\end{aligned}$$

From before, we know that the natural frequencies and mass-normalized modal functions are, respectively:

$$\omega_j = j\pi \sqrt{\frac{T}{\rho L^2}}$$

$$\tilde{\phi}^{(j)} = \sqrt{\frac{2}{\rho L}} \sin j\pi \frac{x}{L}$$

Therefore,

$$\begin{aligned}\hat{F}_j &= \int_0^L \tilde{\phi}^{(j)}(x) F(x)\, dx \\ &= \sqrt{\frac{2}{\rho L}} \int_0^L F(x) \sin j\pi \frac{x}{L}\, dx\end{aligned}$$

$F(x) = f_0 = \text{constant for (a)}$

$$
\begin{aligned}
\hat{F}_j &= \sqrt{\frac{2}{\rho L}}\ f_0 \int_0^L \sin j\pi \frac{x}{L}\, dx \\
&= \sqrt{\frac{2}{\rho L}}\ f_0 \left(-\frac{L}{j\pi}\right) \left[\cos j\pi \frac{x}{L}\right]_{x=0}^{x=L} \\
&= \sqrt{\frac{2}{\rho L}}\ f_0 \left(\frac{L}{j\pi}\right) \left[1 - \cos j\pi\right] \\
&= \begin{cases} 0 & ; \quad j = 2,4,6,\ldots \\ \dfrac{f_0}{j\pi}\sqrt{\dfrac{8L}{\rho}} & ; \quad j = 1,3,5,\ldots \end{cases}
\end{aligned}
$$

Therefore, from above:

$$
\begin{aligned}
u_p(x,t) &= \left\{ \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x)\, \frac{\hat{F}_j / \omega_j^2}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\
&= \left\{ \sqrt{\frac{8L}{\rho}} f_0 \sum_{j=odd} \frac{\tilde{\phi}^{(j)}(x)}{j\pi}\, \frac{1/\omega_j^2}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\
&= \left\{ \sqrt{\frac{8L}{\rho}} f_0 \sqrt{\frac{2}{\rho L}} \sum_{j=odd} \frac{\sin j\pi x/L}{j\pi}\, \frac{\rho L^2 / (j\pi)^2 T}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\
&= \left\{ \frac{4 f_0 L^2}{T} \sum_{j=odd} \frac{\sin j\pi x/L}{(j\pi)^3}\, \frac{1}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\
&= U(x) \sin \Omega t
\end{aligned}
$$

where

$$U(x) = \sum_{j=odd} B_j(x)\, H(\Omega/\omega_j)$$

$$B_j(x) = \frac{4 f_0 L^2}{T} \frac{\sin j\pi x/L}{(j\pi)^3}$$

$$H(\Omega/\omega_j) = \frac{1}{1 - \Omega^2/\omega_j^2}$$

$\underline{F(x) \text{ as shown in figure for (b)}}$

$$
\begin{aligned}
\hat{F}_j &= \sqrt{\frac{2}{\rho L}} \int_0^L F(x) \sin j\pi \frac{x}{L}\, dx \\
&= \sqrt{\frac{2}{\rho L}}\; f_0 \int_{L/2}^{L} \sin j\pi \frac{x}{L}\, dx \\
&= \sqrt{\frac{2}{\rho L}}\; f_0 \left(-\frac{L}{j\pi} \right) \left[\cos j\pi \frac{x}{L} \right]_{x=L/2}^{x=L} \\
&= \sqrt{\frac{2}{\rho L}}\; f_0 \left(\frac{L}{j\pi} \right) \left[\cos \frac{j\pi}{2} - \cos j\pi \right] \\
&= \begin{cases} 0 & ; \quad j = 4,8,12,\ldots \\ \sqrt{\dfrac{2}{\rho L}}\; f_0 \left(\dfrac{L}{j\pi} \right) \left[\cos \dfrac{j\pi}{2} - \cos j\pi \right] & ; \quad j = 1,3,5,\ldots \end{cases}
\end{aligned}
$$

Therefore, from above:

$$
u_p(x,t) = \left\{ \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x)\, \frac{\hat{F}_j / \omega_j^2}{1 - \Omega^2 / \omega_j^2} \right\} \sin \Omega t = U(x) \sin \Omega t
$$

where

$$
U(x) = \sum_{j=1}^{\infty} B_j(x)\, H(\Omega / \omega_j)
$$

$$
B_j(x) = 2 f_0 L^2 \left[\frac{\cos \frac{j\pi}{2} - \cos j\pi}{(j\pi)^3} \right] \sin j\pi x / L
$$

$$
H(\Omega / \omega_j) = \frac{1}{1 - \Omega^2 / \omega_j^2}
$$

$\underline{F(x) = f_0 \delta(x-a) \text{ for (d)}}$

$$\begin{aligned}\hat{F}_j &= \sqrt{\frac{2}{\rho L}}\ f_0 \int_0^L \delta(x-a) \sin j\pi \frac{x}{L}\, dx \\ &= \sqrt{\frac{2}{\rho L}}\ f_0 \sin j\pi \frac{a}{L}\end{aligned}$$

Therefore, from above:

$$\begin{aligned}u_p(x,t) &= \left\{ \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x)\ \frac{\hat{F}_j / \omega_j^2}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\ &= \left\{ \sqrt{\frac{2}{\rho L}} f_0 \sum_{j=1}^{\infty} \tilde{\phi}^{(j)}(x) \sin j\pi \frac{a}{L} \frac{1/\omega_j^2}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\ &= \left\{ \sqrt{\frac{2}{\rho L}} f_0 \sqrt{\frac{2}{\rho L}} \sum_{j=1}^{\infty} (\sin j\pi x/L) \left(\sin j\pi \frac{a}{L} \right) \left(\frac{\rho L^2 / (j\pi)^2 T}{1 - \Omega^2/\omega_j^2} \right) \right\} \sin \Omega t \\ &= \left\{ \frac{2 f_0 L}{T} \sum_{j=1}^{\infty} \left(\frac{\sin j\pi x/L}{(j\pi)^2} \right) \left(\sin j\pi \frac{a}{L} \right) \frac{1}{1 - \Omega^2/\omega_j^2} \right\} \sin \Omega t \\ &= U(x) \sin \Omega t\end{aligned}$$

where

$$U(x) = \sum_{j=1}^{\infty} B_j(x)\ H(\Omega/\omega_j)$$

$$B_j(x) = \frac{2 f_0 L}{T} \left(\frac{\sin j\pi x/L}{(j\pi)^2} \right) \left(\sin j\pi \frac{a}{L} \right)$$

$$H(\Omega/\omega_j) = \frac{1}{1 - \Omega^2/\omega_j^2}$$

Summary

We have seen that the general form of the solution for problems a) – d) above can be written in the general form of

$$u_p(x,t) = U(x)\sin\Omega t = \left\{ \sum_{j=1}^{\infty} B_j(x)\, H(\Omega/\omega_j) \right\} \sin\Omega t$$

As in the discrete model case, we see that the forced response is a linear combination of the modal functions with the contribution of each mode depending on the excitation frequency. This also corresponds to a *synchronous* motion, with the "shape" of the response, *U(x),* not changing with time. For a given value of x, the frequency response function can be sketched as in the discrete models (see last example).

The difference among the four string examples studied here lies in the contribution of the modal functions to these functions $B_j(x)$ making up *U(x)*. In each case, there are some modes that make no contribution, and the contribution of the participating modes decreasing with higher mode numbers. Considering problems a) – d) individually:

a) Here, only the odd-number modes make a contribution to the response. Can you see why the even-numbered modes do not add to the response? The contribution of the odd-numbered modes decreases with the cube of the mode number, $1/j^3$.

b) For this problem, mode numbers 4, 8, 12, 16,… do not contribute to the response. Can you see why this is true? The contribution of the participating modes decreases with the cube of the mode number, $1/j^3$.

c) This is the problem that you worked out. You should have found the response to be quite simple since very few modes contribute to the response. Can you see why this is true?

d) For the case of the point load, the modes for which $\sin(j\pi a/L) = 0$. That is, if the point load is applied at the node of the jth mode, the jth mode cannot make any contribution to the response. For example, if $a = L/3$, then modes of j = 3,6,9,12,… are not excited. Can you see why this is true?