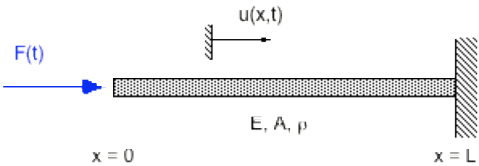
Shock Response to Pulse Loads - 2nd Order Continuous Systems

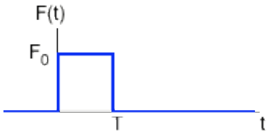
Here we will consider a pulse load on the free end of a free-fixed rod.



EOM for Rod[1]

$$-EA\frac{\partial^2 u}{\partial x^2} + \rho A\frac{\partial^2 u}{\partial t^2} = f(x,t) = \delta(x)F(t) \tag{1}$$

where



Modal EOM's
Using the modal transformation:

$$u(x,t) = \sum_{j=1}^{\infty}\tilde{\phi}^{(j)}(x)q_j(t) \tag{2}$$

produces the following (uncoupled) modal EOM's:

$$\ddot{q}_j + \omega_j^2 q_j = \int_0^L \tilde{\phi}^{(j)} f(x,t)dx = \hat{F}_j(t) \tag{3}$$

where[2]

1 The function δ(x) is the Dirac delta function. See the Help Files page of the course website for details on the Dirac delta function.

$$\omega_j = \frac{2j-1}{2}\pi\sqrt{\frac{E}{\rho L^2}} = \text{natural frequencies} \tag{4}$$

$$\tilde{\phi}^{(j)}(x) = \sqrt{\frac{2}{\rho AL}}\cos\frac{2j-1}{2}\pi\frac{x}{L} = \text{mass normalized modal functions} \tag{5}$$

$$\begin{aligned}\hat{F}_j(t) &= \int_0^L \tilde{\phi}^{(j)} f(x,t)dx \\ &= \left[\int_0^L \tilde{\phi}^{(j)}(x)\delta(x)dx\right]F(t) \\ &= \tilde{\phi}^{(j)}(0)F(t)\end{aligned} \tag{6}$$

Solution of Modal EOM's

For zero initial conditions, the response of each modal equations is given by[3]:

$$\begin{aligned}q_j(t) &= \frac{\tilde{\phi}^{(j)}(0)}{\omega_j}\int_0^t g(\tau)\sin\omega_j(t-\tau)d\tau \\ &= \begin{cases}\dfrac{F_0\tilde{\phi}^{(j)}(0)}{\omega_j^2}\left[1-\cos 2\pi\dfrac{t}{T_j}\right] & ; \quad t < T \\[2ex] \dfrac{F_0\tilde{\phi}^{(j)}(0)}{\omega_j^2}\left[\cos 2\pi\left(\dfrac{t-T}{T_j}\right)-\cos 2\pi\dfrac{t}{T_j}\right] & ; \quad t > T\end{cases}\end{aligned} \tag{7}$$

where $T_j = 2\pi/\omega_j$.

Total Solution of Rod

Using (2), (4), (5) and (7) gives:

$$u(x,t) = \sum_{j=1}^{\infty}\tilde{\phi}^{(j)}(x)q_j(t) \tag{8}$$

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2 As you will see on the Help Files page of the course website, one property of the Dirac delta function δ(x-a) is:

$$\int_0^L g(x)\delta(x-a)dx = g(a)$$

3 Review the results from the single-DOF system response to rectangular pulses.

Shown below is the distribution of displacements in the rod at a given instant in time. The wave form shown here initially travels to the right, reflects off the wall, travels to the left, reflects off the free end, travels to the right, ...

In this motion, the shape of the deformation remains constant, with the wave traveling as described above.