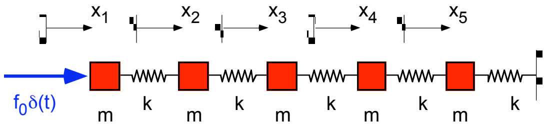
Shock Response to General Pulse Loads - Multi DOF Systems



EOM for System

$$[M]\ddot{\underline{x}} + [K]\underline{x} = \underline{f}_0\delta(t) \tag{1}$$

where

$$[M] = m[I]$$

$$[K] = \begin{bmatrix} k & -k & & & \\ -k & 2k & -k & & \\ & -k & 2k & -k & \\ & & -k & 2k & -k \\ & & & -k & 2k \end{bmatrix}$$

$$\underline{f}_0 = \{f_0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$

Modal EOM's

Using the modal transformation:

$$\underline{x}(t) = \sum_{j=1}^{5} \underline{\tilde{\phi}}^{(j)} q_j(t) \tag{2}$$

produces the following (uncoupled) modal EOM's:

$$\ddot{q}_j + \omega_j^2 q_j = \hat{F}_j(t) \tag{3}$$

where

$$\omega_j = \text{natural frequencies} \tag{4}$$

$$\underline{\tilde{\phi}}^{(j)} = \text{mass normalized modal vectors} \tag{5}$$

$$\hat{F}_j(t) = \underline{\tilde{\phi}}^{(j)T}\,\underline{f}_0\,\delta(t) = \tilde{\phi}_1^{(j)} f_0\,\delta(t) \tag{6}$$

Solution of Modal EOM's
For zero initial conditions, the response of each modal equations is given by:

$$q_j(t) = \frac{\tilde{\phi}_1^{(j)} f_0}{\omega_j}\int_0^t \delta(\tau)\sin\omega_j(t-\tau)\,d\tau = \frac{\tilde{\phi}_1^{(j)} f_0}{\omega_j}\sin\omega_j t \tag{7}$$

Total Solution of System

Using (2), (4), (5) and (7) gives:

$$\begin{aligned}\underline{x}(t) &= \sum_{j=1}^{5}\underline{\tilde{\phi}}^{(j)} q_j(t)\\ &= f_0\sum_{j=1}^{5}\underline{\tilde{\phi}}^{(j)}\,\frac{\tilde{\phi}_1^{(j)}}{\omega_j}\sin\omega_j t\end{aligned} \tag{8}$$

or

$$\begin{aligned}x_p(t) &= \sum_{j=1}^{5}\tilde{\phi}_p^{(j)} q_j(t)\\ &= f_0\sum_{j=1}^{5}\frac{\tilde{\phi}_p^{(j)}\,\tilde{\phi}_1^{(j)}}{\omega_j}\sin\omega_j t\end{aligned} \tag{9}$$

The force transmitted by the spring between the pth and (p+1)th springs for p = 1, 2, 3, 4 is given by:

$$\begin{aligned}F_p(t) &= k\left[x_{p+1}(t) - x_p(t)\right]\\ &= k\,f_0\left[\sum_{j=1}^{5}\left(\tilde{\phi}_{p+1}^{(j)} - \tilde{\phi}_p^{(j)}\right)\frac{\tilde{\phi}_1^{(j)}}{\omega_j}\sin\omega_j t\right]\end{aligned} \tag{10}$$

Numerical Results
Solving for the five natural frequencies and mass-normalized modal vectors gives:

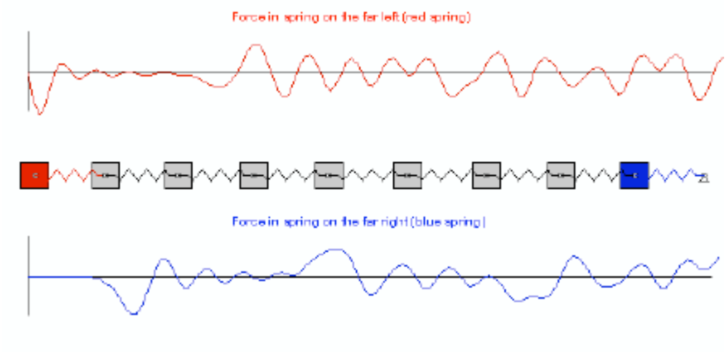
$$
\begin{aligned}
\omega_1 &= 0.2846\sqrt{k/m} \\
\omega_2 &= 0.8308\sqrt{k/m} \\
\omega_3 &= 1.3097\sqrt{k/m} \\
\omega_4 &= 1.6825\sqrt{k/m} \\
\omega_5 &= 1.9190\sqrt{k/m}
\end{aligned}
\tag{11}
$$

$$
\left[\tilde{P}\right] = \left[\underline{\tilde{\phi}}^{(1)} \quad \underline{\tilde{\phi}}^{(2)} \quad \underline{\tilde{\phi}}^{(3)} \quad \underline{\tilde{\phi}}^{(4)} \quad \underline{\tilde{\phi}}^{(5)}\right]
= \begin{bmatrix}
0.5969 & -0.5485 & 0.4557 & -0.3260 & -0.1699 \\
0.5485 & -0.1699 & -0.3260 & 0.5969 & 0.4557 \\
0.4557 & 0.3260 & -0.5485 & -0.1699 & -0.5969 \\
0.3260 & 0.5969 & 0.1699 & -0.4557 & 0.5485 \\
0.1699 & 0.4557 & 0.5969 & 0.5485 & -0.3260
\end{bmatrix} \frac{1}{\sqrt{m}}
\tag{12}
$$

With these results, we can write:

$$
\begin{aligned}
F_p(t) &= k f_0 \left[\sum_{j=1}^{5} \left(\tilde{\phi}_{p+1}^{(j)} - \tilde{\phi}_p^{(j)} \right) \frac{\tilde{\phi}_1^{(j)}}{\omega_j} \sin \omega_j t \right] \\
&= \sqrt{\frac{k}{m}}\, f_0 \left[\begin{Bmatrix} -0.1014 \\ -0.1946 \\ -0.2720 \\ -0.3274 \end{Bmatrix} \sin \omega_1 t + \begin{Bmatrix} -0.2500 \\ -0.3274 \\ -0.1788 \\ 0.0932 \end{Bmatrix} \sin \omega_2 t + \begin{Bmatrix} -0.2720 \\ -0.0774 \\ 0.2500 \\ 0.1486 \end{Bmatrix} \sin \omega_3 t \right. \\
&\qquad \left. + \begin{Bmatrix} -0.1788 \\ 0.1486 \\ 0.0554 \\ -0.1946 \end{Bmatrix} \sin \omega_4 t + \begin{Bmatrix} -0.0554 \\ 0.0932 \\ -0.1014 \\ 0.0774 \end{Bmatrix} \sin \omega_5 t \right]
\end{aligned}
$$

Shown below is the force in two springs for an equivalent nine-DOF model. Here we see that the deformation in the spring to the left responds immediately to the impact load. There is a time delay before the spring on the far right responds indicating that it takes a finite time for the wave to travel along the system. Note that the shape of the response is not preserved (the motion is NOT periodic). This is in contrast with the shock response of the rod (also presented as an example in this lecture) for which the shape of the traveling wave is preserved in time.



Force in spring on the far left (red spring)
Force in spring on the far right (blue spring)