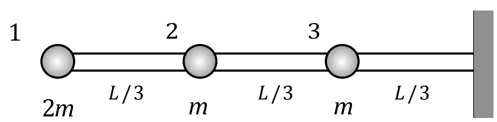
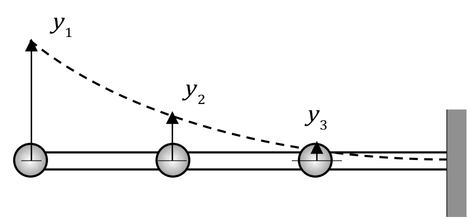
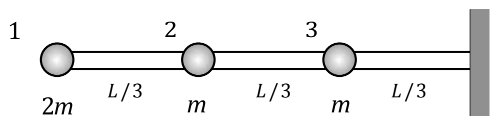
Example A5.3

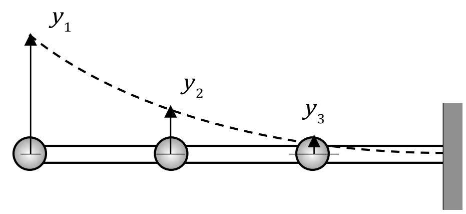
Given: Three particles of mass m are attached to a thin beam of length L, with the beam having mass that is negligible compared to the mass of the particles. Let y_1, y_2 and y_3 represent the transverse displacements of particles 1, 2 and 3, respectively.

Find: For this problem:
a) Write down the dynamical matrix $[D]$ for the system.
b) Use the dynamical matrix to produce a lower bound on the lowest natural frequency of the system.
c) Use the Rayleigh method to produce an upper bound on the lowest natural frequency of the system. Use the static deformation due to weight as your trial function. In doing so, write down the Rayleigh quotient using the flexibility matrix (not the stiffness matrix); that is, do so in way that you do not need to invert the flexibility matrix to find the stiffness matrix.
d) Use the power method to produce the lowest natural frequency for the system. Carry out enough iterations such that your result is accurate to four digits to the right of the decimal. Verify that your result is between the bounds determined in a) and b) above.









SOLUTION

From the solution for Example I.6.1, the flexibility matrix is given by:

$$[A]=\frac{1}{162}\begin{bmatrix} 54 & 28 & 8 \\ 28 & 16 & 5 \\ 8 & 5 & 2 \end{bmatrix}\frac{L^3}{EI}$$

From the following kinetic energy expression:

$$T=\frac{1}{2}(2m)\dot{y}_1^2+\frac{1}{2}m\dot{y}_2^2+\frac{1}{2}m\dot{y}_2^2$$

we see that the mass matrix for this system is:

$$[M]=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}m$$

Therefore, the dynamical matrix for the system is:

$$[D]=[A][M]=\frac{1}{162}\begin{bmatrix}54 & 28 & 8\\ 28 & 16 & 5\\ 8 & 5 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\frac{mL^3}{EI}=\frac{1}{162}\begin{bmatrix}108 & 28 & 8\\ 56 & 16 & 5\\ 16 & 5 & 2\end{bmatrix}\frac{mL^3}{EI}$$

Lower bound

From this, we have:

$$tr[D]=(108+16+2)\frac{mL^3}{162EI}=0.778\frac{mL^3}{EI}$$

Therefore,

$$\omega_1>\frac{1}{\sqrt{tr[D]}}=1.1337\sqrt{\frac{EI}{mL^3}}$$

Upper bound

The static deformation of the beam due to weight of the masses is:

$$\vec{y}_{st}=[K]^{-1}\vec{W}=[A]\vec{W}=\frac{1}{162}\begin{bmatrix}54 & 28 & 8\\ 28 & 16 & 5\\ 8 & 5 & 2\end{bmatrix}\begin{Bmatrix}2mg\\ mg\\ mg\end{Bmatrix}\frac{L^3}{EI}=\frac{1}{162}\begin{Bmatrix}144\\ 77\\ 23\end{Bmatrix}\frac{mgL^3}{EI}$$

The Rayleigh quotient using the static deformation as the trial vector is:

$$Q(\vec{y}_{st})=\frac{\vec{y}_{st}^T[K]\vec{y}_{st}}{\vec{y}_{st}^T[M]\vec{y}_{st}}=\frac{([A]\vec{W})^T[K]([A]\vec{W})}{\vec{y}_{st}^T[M]\vec{y}_{st}}=\frac{\vec{W}^T[A][K][A]\vec{W}}{\vec{y}_{st}^T[M]\vec{y}_{st}}=\frac{\vec{W}^T[A]\vec{W}}{\vec{y}_{st}^T[M]\vec{y}_{st}}$$

$$=\frac{\begin{Bmatrix}2\\ 1\\ 1\end{Bmatrix}^T\begin{bmatrix}54 & 28 & 8\\ 28 & 16 & 5\\ 8 & 5 & 2\end{bmatrix}\begin{Bmatrix}2\\ 1\\ 1\end{Bmatrix}(mg)^2\frac{L^3}{162EI}}{\begin{Bmatrix}144\\ 77\\ 23\end{Bmatrix}^T\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}144\\ 77\\ 23\end{Bmatrix}m(mg)^2\left(\frac{L^3}{162EI}\right)^2}=1.3114\frac{EI}{mL^3}$$

Therefore,

$$\omega_1\le\sqrt{Q(\vec{y}_{st})}=\sqrt{1.3114\frac{EI}{mL^3}}=1.1452\sqrt{\frac{EI}{mL^3}}$$

Power method

Choose the following trial vector:

$$\vec{v}^{(0)}=\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}$$

Using the following iteration, we arrive at an estimate of the first natural frequency and modal vector:

$$\vec{v}^{(1)} = [D]\vec{v}^{(0)} = [D]\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.8889 \\ 0.4753 \\ 0.1420 \end{Bmatrix}\frac{mL^3}{EI} = 0.8889\begin{Bmatrix} 1 \\ 0.5347 \\ 0.1597 \end{Bmatrix}\frac{mL^3}{EI}$$

$$\vec{v}^{(2)} = [D]\vec{v}^{(1)} = [D]\begin{Bmatrix} 1 \\ 0.5347 \\ 0.1597 \end{Bmatrix} = \begin{Bmatrix} 0.7670 \\ 0.4034 \\ 0.1172 \end{Bmatrix}\frac{mL^3}{EI} = 0.7670\begin{Bmatrix} 1 \\ 0.5260 \\ 0.1529 \end{Bmatrix}\frac{mL^3}{EI}$$

$$\vec{v}^{(3)} = [D]\vec{v}^{(2)} = [D]\begin{Bmatrix} 1 \\ 0.5260 \\ 0.1529 \end{Bmatrix} = \begin{Bmatrix} 0.7651 \\ 0.4023 \\ 0.1169 \end{Bmatrix}\frac{mL^3}{EI} = 0.7651\begin{Bmatrix} 1 \\ 0.5259 \\ 0.1528 \end{Bmatrix}\frac{mL^3}{EI}$$

$$\vec{v}^{(4)} = [D]\vec{v}^{(3)} = [D]\begin{Bmatrix} 1 \\ 0.5259 \\ 0.1528 \end{Bmatrix} = \begin{Bmatrix} 0.7651 \\ 0.4023 \\ 0.1169 \end{Bmatrix}\frac{mL^3}{EI} = 0.7651\begin{Bmatrix} 1 \\ 0.5259 \\ 0.1528 \end{Bmatrix}\frac{mL^3}{EI}$$

Therefore, to four digits to the right of the decimal:

$$\omega_1 = \frac{1}{\sqrt{0.7651mL^3 / EI}} = 1.143\sqrt{\frac{EI}{mL^3}}$$

which lies between the upper and lower bounds found above.