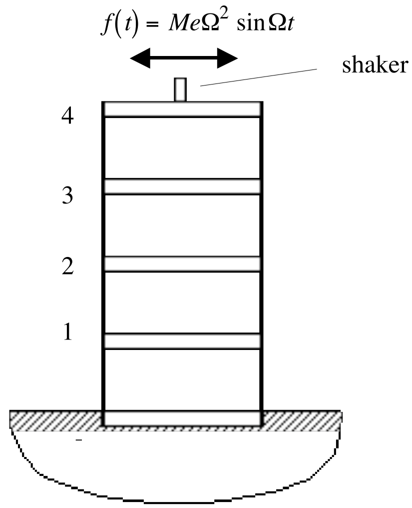
Example A4.21

Here we reconsider the building problem developed earlier in the course. Here we will mount an eccentric shaker on the top floor of the building to simulate a modal test of the building. Find the response of the building as a function of the shaker frequency Ω. How would the response change if the shaker were instead mounted on the second floor from the top?



The natural frequencies and modal vectors are repeated here:

$$\omega_1 = 0.4912\sqrt{k/m} \quad ; \quad \omega_2 = 1.4142\sqrt{k/m}$$
$$\omega_3 = 2.1667\sqrt{k/m} \quad ; \quad \omega_4 = 2.6579\sqrt{k/m}$$

$$\left[\tilde{P}\right] = \left[\underline{\phi}^{(1)}, \ \underline{\phi}^{(2)}, \ \underline{\phi}^{(3)}, \ \underline{\phi}^{(4)}\right]$$
$$= \frac{1}{\sqrt{m}}\begin{bmatrix} 0.2280 & 0.5774 & 0.6565 & 0.4285 \\ 0.4285 & 0.5774 & -0.2280 & -0.6565 \\ 0.5774 & 0 & -0.5774 & 0.5774 \\ 0.6565 & -0.5774 & 0.4285 & -0.2280 \end{bmatrix}$$

The EOM's for this model of the building are given by:

$$[M]\,\underline{\ddot{x}} + [K]\,\underline{x} = \underline{f}(t)$$

where:

$$\underline{f}(t) = Me\Omega^2 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \sin\Omega t$$

The uncoupled EOM's become:

$$\ddot{q}_j + \omega_j^2 q_j = \hat{f}_j(t)$$

where $\underline{x}(t) = \sum_{j=1}^{4} \underline{\tilde{\phi}}^{(j)} q_j(t)$ and the modal forcings are given by:

$$\hat{f}_j(t) = \underline{\tilde{\phi}}^{(j)T}\,\underline{f}(t)$$

Explicitly we have:

$$\hat{f}_1(t) = \phi_4^{(1)} Me\Omega^2 \sin\Omega t = \hat{f}_{01} \sin\Omega t$$
$$\hat{f}_2(t) = \phi_4^{(2)} Me\Omega^2 \sin\Omega t = \hat{f}_{02} \sin\Omega t$$
$$\hat{f}_3(t) = \phi_4^{(3)} Me\Omega^2 \sin\Omega t = \hat{f}_{03} \sin\Omega t$$
$$\hat{f}_4(t) = \phi_4^{(4)} Me\Omega^2 \sin\Omega t = \hat{f}_{04} \sin\Omega t$$

Solving for the particular response of the modal EOM's (harmonic excitation):

$$q_{jp}(t) = \frac{\hat{f}_{0j}/\omega_j^2}{1 - \Omega^2/\omega_j^2} \sin\Omega t = Q_j \sin\Omega t$$

where the modal amplitudes of response are:

$$Q_1 = \phi_4^{(1)} Me \frac{\Omega^2}{\omega_1^2} \frac{1}{1-\Omega^2/\omega_1^2} = \phi_4^{(1)} MeH(\Omega/\omega_1)$$

$$Q_2 = \phi_4^{(2)} Me \frac{\Omega^2}{\omega_2^2} \frac{1}{1-\Omega^2/\omega_2^2} = \phi_4^{(2)} MeH(\Omega/\omega_2)$$

$$Q_3 = \phi_4^{(3)} Me \frac{\Omega^2}{\omega_3^2} \frac{1}{1-\Omega^2/\omega_3^2} = \phi_4^{(3)} MeH(\Omega/\omega_3)$$

$$Q_4 = \phi_4^{(4)} Me \frac{\Omega^2}{\omega_4^2} \frac{1}{1-\Omega^2/\omega_4^2} = \phi_4^{(4)} MeH(\Omega/\omega_4)$$

and

$$H(\Omega/\omega_j) = \frac{\Omega^2}{\omega_j^2} \frac{1}{1-\Omega^2/\omega_j^2}$$

The total particular response for the harmonic excitation on the fifth floor is given by:

$$\begin{aligned} \underline{x}(t) &= \sum_{j=1}^{4} \underline{\tilde{\phi}}^{(j)} q_j(t) \\ &= \left\{ \sum_{j=1}^{4} \underline{\tilde{\phi}}^{(j)} Q_j \right\} \sin \Omega t \\ &= \underline{X} \sin \Omega t \end{aligned}$$

where the amplitude of the response of the ith floor is given by:

$$\begin{aligned} X_i &= \sum_{j=1}^{4} \tilde{\phi}_i^{(j)} Q_j \\ &= Me \sum_{j=1}^{4} \tilde{\phi}_i^{(j)} \tilde{\phi}_4^{(j)} H(\Omega/\omega_j) \end{aligned}$$

Remarks:

a) The above expression for $X_i(\Omega)$ shows that the amplitude of response for the ith floor for a given excitation frequency is given directly in terms of the components of the modal vectors $\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}$ and the natural frequencies $H(\Omega/\omega_j)$.

b) At first glance, this expression for $X_i(\Omega)$ looks simple. However, with closer inspection we see that the result is somewhat complicated since it involves a summation over the number of DOF's (in this case, four DOF's) of the terms $\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}H(\Omega/\omega_j)$. Having said this, we can make some sense out of these response functions without relying on the numerical details of this sum. To help us in this, consider the following observations:

   i) Since $H(\Omega/\omega_j)\to 0$ as $\Omega\to 0$ for all modes, we see that $X_i(\Omega)\to 0$ as $\Omega\to 0$ for all floors.

   ii) Since $H(\Omega/\omega_j)\to +\infty$ as $\Omega\to\omega_j^-$ and $H(\Omega/\omega_j)\to -\infty$ as $\Omega\to\omega_j^+$, the magnitude of response $|X_i(\Omega)|\to\infty$ with a sign change at $\Omega=\omega_j$ (i.e., resonance). The sign of $X_i(\Omega)$ at $\Omega\to\omega_j^-$ is the SAME as the sign of $\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}$, and the sign of $X_i(\Omega)$ at $\Omega\to\omega_j^+$ is OPPOSITE of the sign of $\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}$.

   iii) We can use the results of ii) above to help us with some of the details of the response between resonances, as we see below.

c) A table below summarizes the values of $B_{ij}=\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}$ for this example. Recall that "i" refers to the floor on which response is desired and "j" is the mode number. As the following exercise shows, this set of numbers tells the whole story about how the response curves for the system look.

$B_{ij}=\tilde{\phi}_i^{(j)}\tilde{\phi}_4^{(j)}$				
	j = 1	***j = 2***	***j = 3***	***j = 4***
i = 1	0.14968	-0.33339	0.28131	-0.09769
i = 2	0.28131	-0.33339	-0.09769	0.14968
i = 3	0.37906	0	-0.24742	-0.13165
i = 4	0.43099	0.33339	0.18361	0.05198

Example – Sketching the frequency response function for the 2^{nd} floor

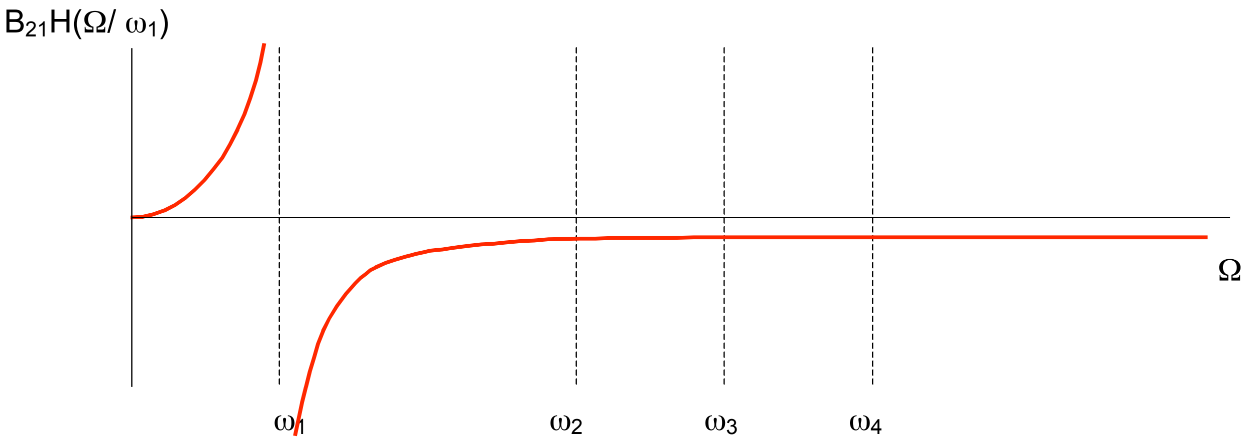
1. Shown to the right are sketches of the functions $B_{2j}H(\Omega/\omega_j)$. Since the table shows that $B_{22}<0$ and $B_{23}<0$, the plots of $B_{22}H(\Omega/\omega_2)$ and $B_{23}H(\Omega/\omega_3)$ are flipped about the Ω-axis, as seen in the figures.
2. To make the sketch of $X_2(\Omega)$, we simply need to add up these four modal contributions.
3. Now take the absolute value of $X_2(\Omega)$ to determine the amplitude of the response.

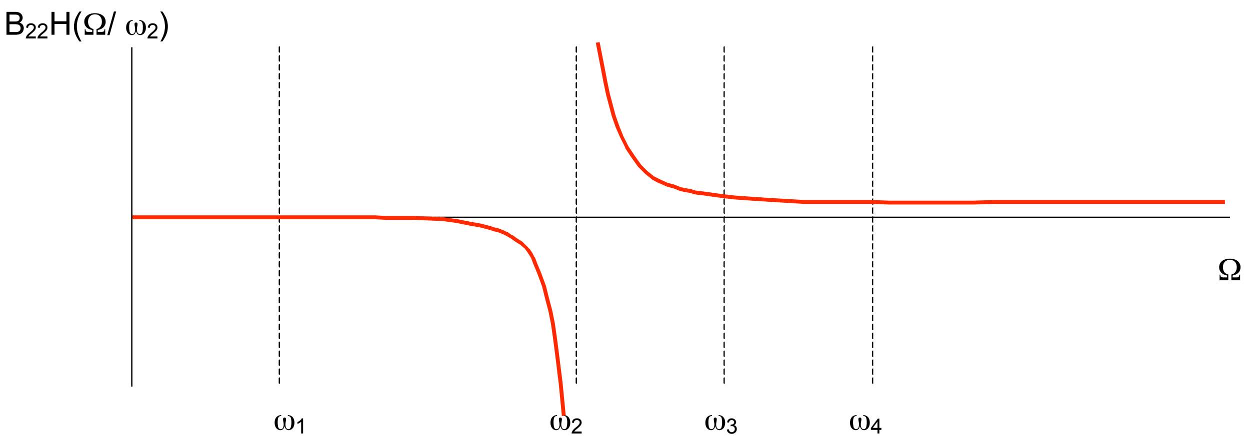
As an exercise, use the above thought process to make sketches of $X_1(\Omega)$, $X_3(\Omega)$ and $X_4(\Omega)$ vs. Ω. Based on these sketches, consider the following questions:

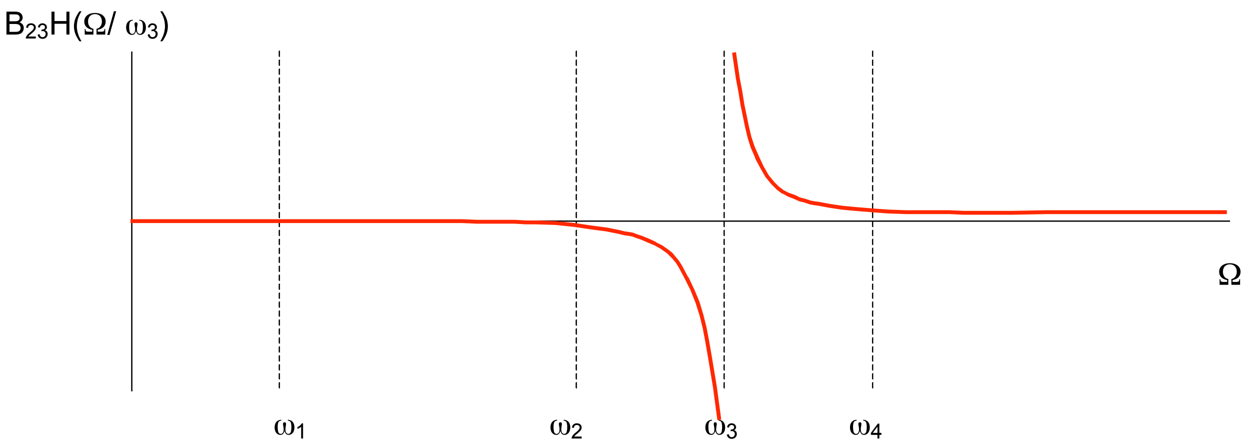
i) How do the plots of $X_1(\Omega)$, $X_3(\Omega)$ and $X_4(\Omega)$ vs. Ω differ qualitatively from those presented here for $X_2(\Omega)$ vs. Ω?

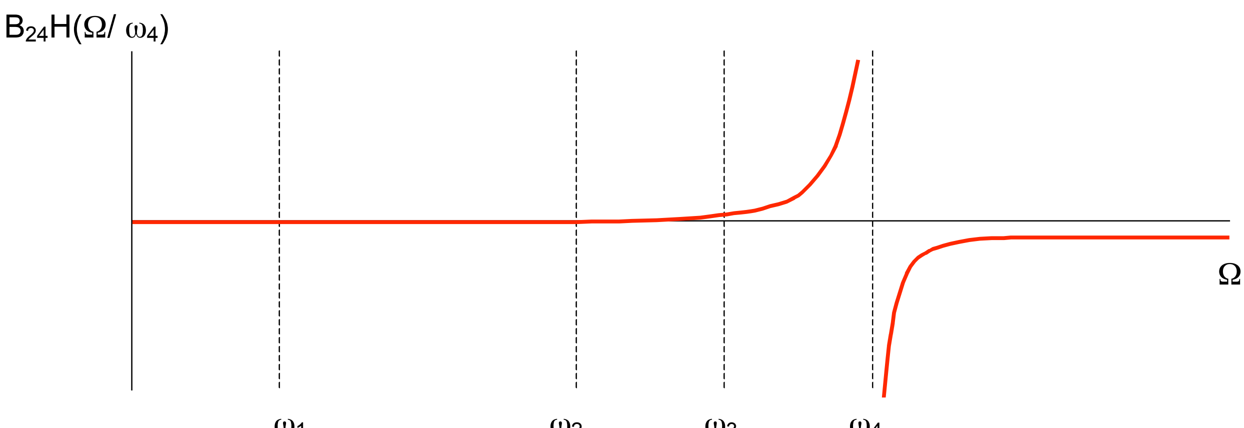
ii) Note that the plot of $X_4(\Omega)$ vs. Ω has an anti-resonance between each resonance. This is an important result to remember, in general. If you measure the response at the point of application of the forcing, you will find an anti-resonance between each resonance. Look back at the equations for the response. Can you see WHY this is true? If not, let's talk.

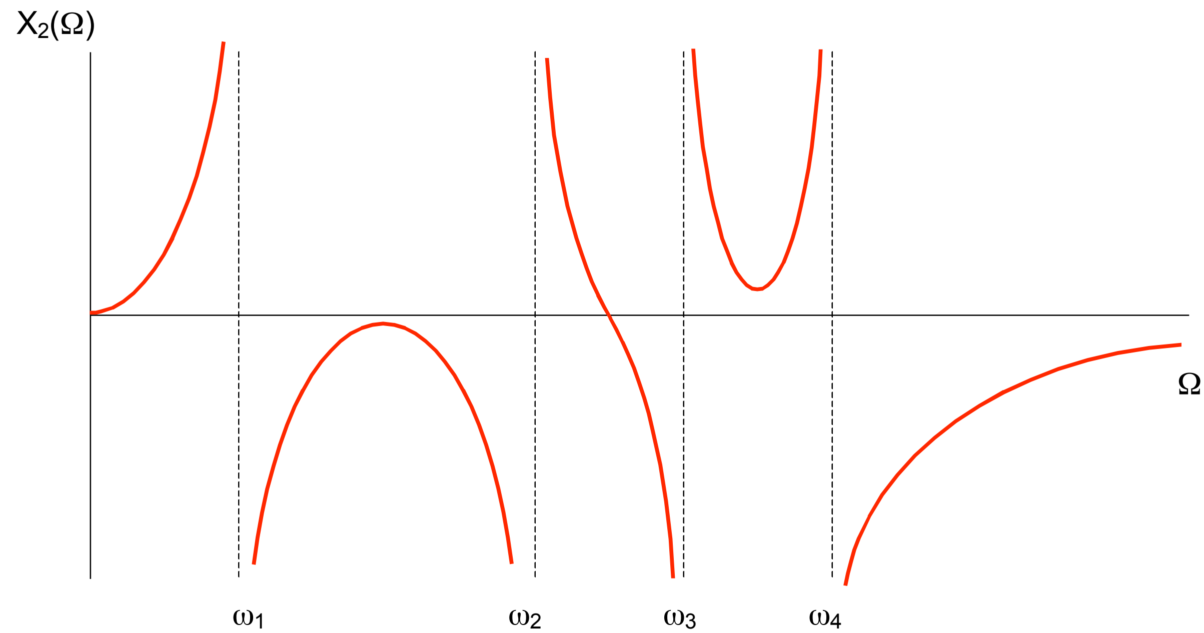
iii) If the forcing is moved from the 5^{th} floor to the 4^{th} floor, how does the response curves change? Why will there not be a resonance at $\Omega=\omega_2$? Look back at the mode shapes to answer this question.


B21H(Ω/ ω1)
Ω
ω1
ω2
ω3
ω4

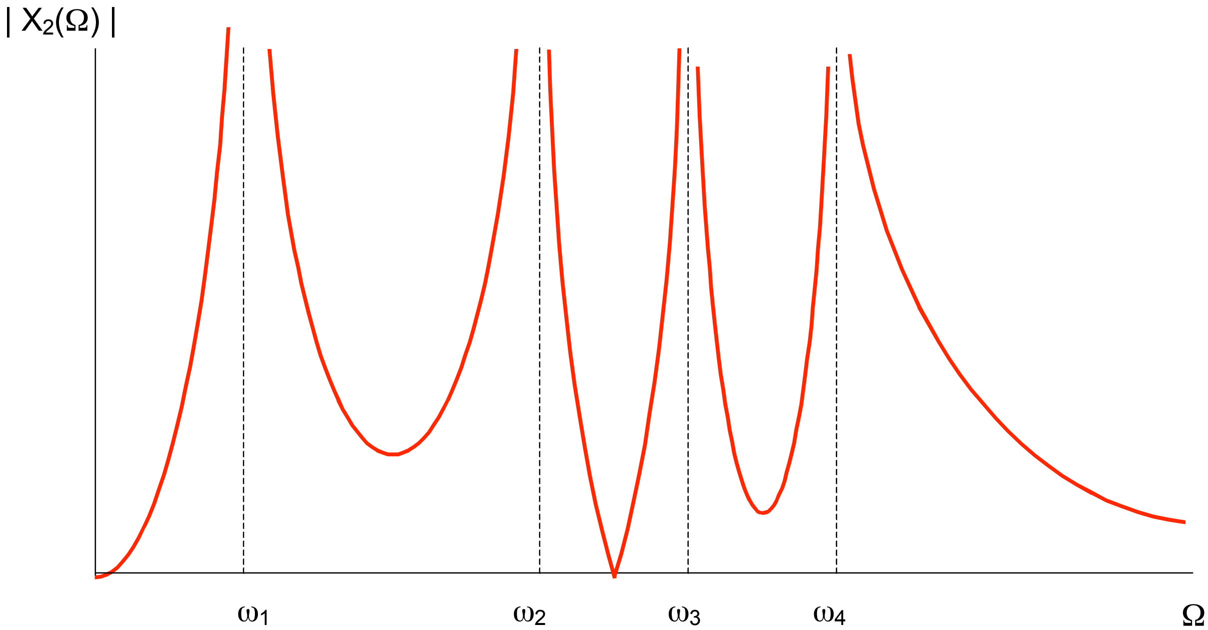

B22H(Ω/ ω2)
Ω
ω1
ω2
ω3
ω4


B23H(Ω/ ω3)
Ω
ω1
ω2
ω3
ω4


B24H(Ω/ ω4)
Ω
ω1
ω2
ω3
ω4



$X_2(\Omega)$
Ω
ω_1
ω_2
ω_3
ω_4



$| X_2(\Omega) |$
ω_1
ω_2
ω_3
ω_4
Ω