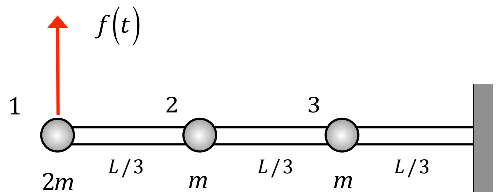
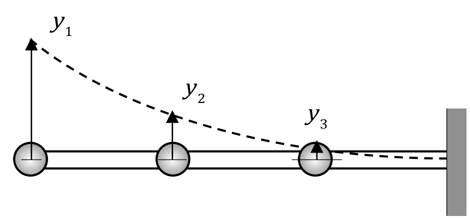
Example A4.20

Given: A point load acts on end particle "1" of the beam: $f(t) = I_F \delta(t)$, where I_F is the impulse of the force $f(t)$. The mass of the beam is negligible as compared to the mass of the three particles.

Find: For this problem:
a) Write down the modal EOMs for the system in symbolic form.
b) Solve the modal equations corresponding to zero initial conditions, again leaving your answer in symbolic form.
c) Write a Matlab (or equivalent) code to find the natural frequencies (scaled by $\sqrt{EI/mL^3}$) and the mass-normalized modal vectors.
d) Based the results in b) and c), comment on the relative size contributions of the three modes to the response.
e) Add to your Matlab code the ability to produce plots of three particle displacements scaled by $I_F/\sqrt{mEI/L^3}$ vs. non-dimensional time, $t\sqrt{EI/ml^3}$.





SOLUTION

Modal EOMs and solutions

The system EOMs are written as:

$$[M]\ddot{\vec{y}}+[K]\vec{y}=\vec{f}(t)=\left\{\begin{array}{c} I_F \\ 0 \\ 0 \end{array}\right\}\delta(t)=\vec{f}_0\delta(t)$$

where:

$$[M]=m\begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}=m[M_0]$$

$$[K]=[A]^{-1}=\frac{EI}{L^3}\left(162\begin{bmatrix}54 & 28 & 8\\ 28 & 16 & 5\\ 8 & 5 & 2\end{bmatrix}^{-1}\right)=\frac{EI}{L^3}[K_0]$$

Write down the following coordinate transformation:

$$\vec{y}(t)=\sum_{j=1}^{3}\hat{\vec{Y}}^{(j)}p_j(t)$$

where $\hat{\vec{Y}}^{(j)}$; $j=1,2,3$ are the mass-normalized modal vectors. Substitution of this transformation into the EOMs and pre-multiplying by $\hat{\vec{Y}}^{(k)T}$ gives the following modal EOMs:

$$\ddot{p}_k+\omega_k^2 p_k=\hat{\vec{Y}}^{(k)T}\vec{f}_0\delta(t)=\hat{Y}_1^{(k)}I_k\delta(t)$$

Solving the above modal EOMs with zero initial conditions gives (through the use of the convolution integral approach):

$$p_k(t)=\frac{\hat{Y}_1^{(k)}I_F}{\omega_k}\int_0^t \sin\omega_k(t-\tau)\delta(\tau)\,d\tau=\frac{\hat{Y}_1^{(k)}I_F}{\omega_k}\sin\omega_k t$$

Therefore, we have:

$$\vec{y}(t)=\sum_{j=1}^{3}\hat{\vec{Y}}^{(j)}p_j(t)=I_F\sum_{j=1}^{3}\frac{\hat{Y}_1^{(j)}}{\omega_j}\hat{\vec{Y}}^{(j)}\sin\omega_j t$$

<u>Natural frequencies and modal vectors</u>

$$\left[-\omega^2[M]+[K]\right]\vec{Y}=\vec{0}$$

$$\left[-m\omega^2[M_0]+\frac{EI}{L^3}[K_0]\right]\vec{Y}=$$

$$\left[-\frac{mL^3}{EI}\omega^2[M_0]+[K_0]\right]\vec{Y}=$$

$$\left[-\mu^2[M_0]+[K_0]\right]\vec{Y}=$$

where $\mu=\sqrt{\frac{mL^3}{EI}}\omega$.

The non-dimensional natural frequencies μ_j ; $j = 1,2,3$ and the mass normalized modal vectors are found using the following Matlab code:

```
clear

A0=[54,28,8;28,16,5;8,5,2]/162;
M0=diag([2;1;1]);
K0=162*inv(A0);

[v,d]=eig(K0,M0);
mu=sqrt(diag(d));
[mu,id]=sort(mu);
v=v(:,id);
alpha=sqrt(diag(v'*M0*v))';
v=v./[ones(3,1)*alpha];
```

The results are:

$$\mu_1 = 14.55 \quad \Rightarrow \quad \omega_1 = 14.55\sqrt{EI / mL^3}$$

$$\mu_2 = 119.95 \quad \Rightarrow \quad \omega_2 = 119.95\sqrt{EI / mL^3}$$

$$\mu_3 = 337.78 \quad \Rightarrow \quad \omega_3 = 337.78\sqrt{EI / mL^3}$$

$$\hat{\vec{Y}}^{(1)} = \begin{Bmatrix} 0.6594 \\ 0.3467 \\ 0.1007 \end{Bmatrix} \frac{1}{\sqrt{m}} \; ; \; \hat{\vec{Y}}^{(2)} = \begin{Bmatrix} 0.2403 \\ -0.7494 \\ -0.5682 \end{Bmatrix} \frac{1}{\sqrt{m}} \; ; \; \hat{\vec{Y}}^{(3)} = \begin{Bmatrix} 0.0859 \\ -0.5641 \\ 0.8167 \end{Bmatrix} \frac{1}{\sqrt{m}}$$

<u>Response</u>

Let

$$\hat{t} = t\sqrt{EI / mL^3}$$

$$z_j = \frac{y_j}{I_F\sqrt{L^3 / mEI}}$$

$$\hat{\vec{Y}}^{(j)} = \frac{1}{\sqrt{m}} \tilde{\vec{Y}}^{(j)}$$

Therefore:

$$\vec{y}(\hat{t}) = I_F\sqrt{\frac{mL^3}{EI}} \left(\frac{1}{\sqrt{m}} \right)^2 \sum_{j=1}^{3} \frac{\tilde{Y}_1^{(j)}}{\mu_j} \tilde{\vec{Y}}^{(j)} \sin \mu_j \hat{t}$$

$$= I_F\sqrt{\frac{L^3}{mEI}} \sum_{j=1}^{3} \frac{\tilde{Y}_1^{(j)}}{\mu_j} \tilde{\vec{Y}}^{(j)} \sin \mu_j \hat{t} \quad \Rightarrow$$

$$\vec{z}(\hat{t}) = \sum_{j=1}^{3} \frac{\tilde{Y}_1^{(j)} \tilde{\vec{Y}}^{(j)}}{\mu_j} \sin \mu_j \hat{t}$$

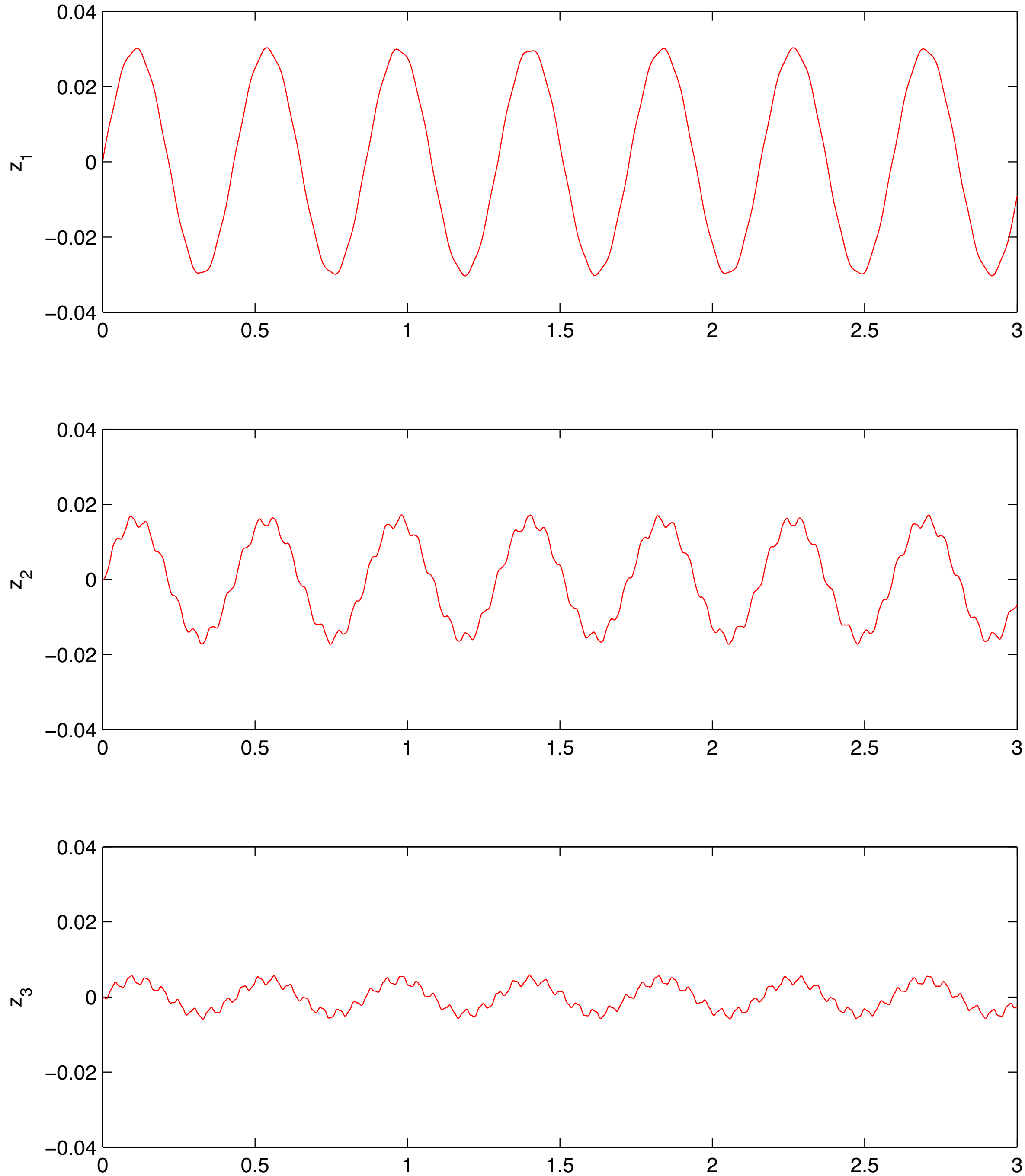
$$= \frac{\tilde{Y}_1^{(1)} \tilde{\vec{Y}}^{(1)}}{\mu_1} \sin \mu_1 \hat{t} + \frac{\tilde{Y}_1^{(2)} \tilde{\vec{Y}}^{(2)}}{\mu_2} \sin \mu_2 \hat{t} + \frac{\tilde{Y}_1^{(3)} \tilde{\vec{Y}}^{(3)}}{\mu_3} \sin \mu_3 \hat{t}$$

$$= \left(\begin{Bmatrix} 29.9 \\ 15.7 \\ 4.56 \end{Bmatrix} \times 10^{-3} \right) \sin 14.55\hat{t} + \left(\begin{Bmatrix} 0.481 \\ -1.50 \\ -1.13 \end{Bmatrix} \times 10^{-3} \right) \sin 119.95\hat{t}$$

$$+ \left(\begin{Bmatrix} 21.8 \\ -143.5 \\ 207.7 \end{Bmatrix} \times 10^{-6} \right) \sin 337.78\hat{t}$$

From this, we see that the first mode dominates the response.

Plots of the response are shown on the next page.


0.04
0.02
0
−0.02
−0.04
z1
z2
z3
0
0.5
1
1.5
2
2.5
3