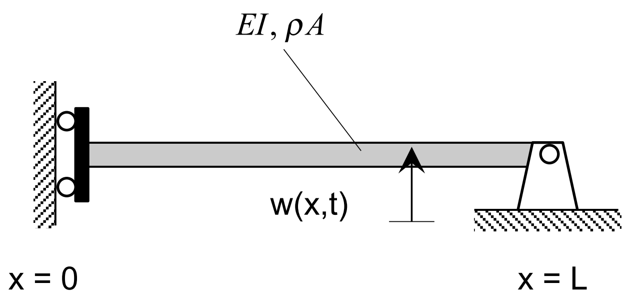
Example A4.19

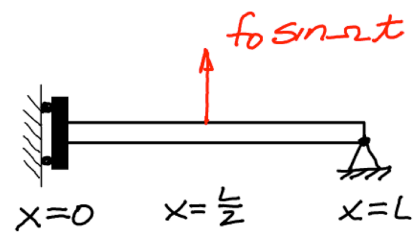
Given: Consider the bending beam shown below with a concentrated harmonic force acting at the mid-length point C on the beam.

Find: For this problem:

a) Use the modal uncoupling approach to determine the particular solution $w_P(t) = W(x)sin\Omega t$ of the beam's EOM.

b) Make a hand sketch of $|W(x = L/2)|$ vs. Ω for a range of values of Ω covering up through the first four resonances of the response. Provide details on how you arrived at this hand sketch.





<u>SOLUTION</u>

$$W(x) = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$$

w/ $\beta^4 = \omega^2 \frac{\rho A}{EI}$

$$\left.\begin{array}{l} \bullet\ \frac{dW}{dx}(0) = 0 = \beta\,[b+d] \\ \bullet\ EI\frac{d^3W}{dx^3}(0) = 0 = EI\beta^3\,[b-d] \end{array}\right\} \text{For } \beta \neq 0 \Rightarrow b = d = 0$$

- $W(L) = 0 = a\cosh\beta L + c\cos\beta L$ (1)
- $EI\frac{d^2W}{dx^2}(L) = 0 = EI\beta^2[a\cosh\beta L - c\cos\beta L]$

$\hookrightarrow$ for $\beta \neq 0$: $0 = a\cosh\beta L - c\cos\beta L$ (2)

Add (1) & (2):

$$2a\cosh\beta L = 0$$

Since $\cosh\beta L \neq 0 \Rightarrow a = 0$

$\therefore$ $c\cos\beta L = 0 \Rightarrow$ for $c \neq 0$: $\beta_j L = \frac{2j-1}{2}\pi$

$$\Rightarrow \quad \omega_j = \beta_j^2 \sqrt{\frac{EI}{\rho A}} = \left(\frac{2j-1}{2}\pi\right)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$$\text{\& } \quad W^{(j)}(x) = \cancel{c_j}^{\,1} \cos \beta_j x = \cos \frac{2j-1}{2}\pi \frac{x}{L}$$

(You are allowed to use results above directly from previous homework assignment.)

<u>Forced Response</u>

Forced beam EOM:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad (3)$$

Modal expansion:

$$w(x,t) = \sum_{j=1}^{\infty} W^{(j)}(x)\, p_j(t) \qquad (4)$$

Substitute (4) into (3), multiply by $W^{(k)}(x)$ & $\int_0^L (\cdot)\,dx$:

$$EI \sum_j \overbrace{\left(\int_0^L W^{(k)} \frac{d^4 W^{(j)}}{dx^4} dx\right)}^{=0;\ k \neq j} p_j + \rho A \sum_j \underbrace{\left(\int_0^L W^{(k)} W^{(j)} dx\right)}_{=0;\ k \neq j} \ddot{p}_j = \int_0^L W^{(k)} f(x,t)\, dx$$

$$\left[\rho A \int_0^L W^{(j)2} dx\right] \ddot{p}_j + \left[EI \int_0^L W^{(j)} \frac{d^4 W^{(j)}}{dx^4} dx\right] p_j = \left[f_0 \int_0^L \delta\left(x - \frac{L}{2}\right) W^{(j)} dx\right] \sin \Omega t$$

$$\ddot{p}_j + \underbrace{\left[\frac{EI \int_0^L W^{(j)} \frac{d^4 W^{(j)}}{dx^4} dx}{\rho A \int_0^L W^{(j)2} dx}\right]}_{=\omega_j^2 \text{ (see BVP on page III-47)}} p_j = \left[\frac{f_0 W^{(j)}\left(\frac{L}{2}\right)}{\rho A \int_0^L W^{(j)2} dx}\right] \sin \Omega t$$

or,

$$\ddot{p}_j + \omega_j^2 p_j = \hat{f}_j \sin \Omega t$$

The particular solutions for these modally-uncoupled EOMs are given by:

$$p_{jP}(t) = \frac{\hat{f}_j}{\omega_j^2 - \Omega^2} \sin \Omega t$$

Therefore, the beam response is:

$$\begin{aligned} w_P(x,t) &= \sum_{j=1}^{\infty} W^{(j)}(x) p_{jP}(t) \\ &= \left\{ \sum_{j=1}^{\infty} W^{(j)}(x) \left(\frac{\hat{f}_j}{\omega_j^2 - \Omega^2} \right) \right\} \sin \Omega t \\ &= W(x) \sin \Omega t \end{aligned}$$

From this, we have:

$$\begin{aligned} W\left(\frac{L}{2}\right) &= \sum_{j=1}^{\infty} W^{(j)}\left(\frac{L}{2}\right) \left(\frac{\hat{f}_j}{\omega_j^2 - \Omega^2} \right) \\ &= f_0 \sum_{j=1}^{\infty} \frac{\left[W^{(j)}\left(\frac{L}{2}\right) \right]^2}{\left(\rho A \int_0^L W^{(j)2} \, dx \right)} \left(\frac{1}{\omega_j^2 - \Omega^2} \right) \end{aligned}$$

<u>Comments</u>:

- Resonances will occur for the jth mode when $\Omega = \omega_j$, unless $W^{(j)}(L/2) = 0$.
- Anti-resonances will occur between every pair of resonances since the coefficient of $1/\left(\omega_j^2 - \Omega^2\right)$ in the above sum is positive for all modes.