Example A4.18

Given: Consider the following damped single-DOF oscillator: $m\ddot{x} + c\dot{x} + kx = f(t)$, where $f(t)$ is an arbitrary forcing. It is also known that the system is critically damped.

Find: For this problem, derive the form of the convolution integral solution for this system. Feel free to start with equations (IV.12) and (IV.13) of the lecture book in your derivation.

For critically damped system ($\zeta \equiv 1$):

$$u(t) = e^{-\omega_n t}$$
$$v(t) = t e^{-\omega_n t} = t\,u(t)$$

Therefore:

$$\dot{u}(t) = -\omega_n e^{-\omega_n t} = -\omega_n u(t)$$
$$\dot{v}(t) = u(t) + t\dot{u}(t) = u(t)(1-\omega_n t)$$

and:

$$u(\tau)\dot{v}(\tau) - \dot{u}(\tau)v(\tau) = u^2(1-\omega_n\tau) + \omega_n\tau\, u$$
$$= u^2(\tau) = e^{-2\omega_n\tau}$$
$$u(\tau)v(t) - u(t)v(\tau) = t\,e^{-\omega_n(t+\tau)} - \tau\,e^{-\omega_n(t+\tau)}$$
$$= (t-\tau)\,e^{-\omega_n(t-\tau)}$$

From this, we have:

$$x(t) = a\,u(t) + b\,v(t) + \frac{1}{m}\int_0^t \left[\frac{u(\tau)v(t) - u(t)v(\tau)}{u(\tau)\dot{v}(\tau) - \dot{u}(\tau)v(\tau)}\right] f(\tau)\,d\tau$$
$$= a\,u(t) + b\,v(t) + \frac{1}{m}\int_0^t \frac{(t-\tau)e^{-\omega_n(t-\tau)}}{e^{-2\omega_n(t+\tau)}} f(\tau)\,d\tau$$
$$= a e^{-\omega_n t} + b t e^{-\omega_n t} + \int_0^t h(t-\tau) f(\tau)\,d\tau$$

w/

$$h(t-\tau) = (t-\tau)\,e^{-\omega_n(t-\tau)}$$
$$\left.\begin{aligned} a &= x(0) \\ b &= \dot{x}(0)/\omega_n \end{aligned}\right\}\ \text{as seen in lecture}$$