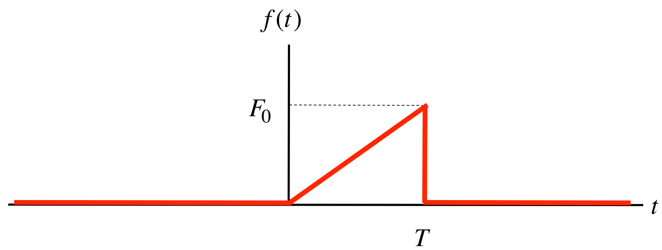
Example A4.17

Given: An undamped oscillator, $m\ddot{x} + kx = f(t)$, has the forcing $f(t)$ shown below. The system has the following initial conditions: $x(0) = \dot{x}(0) = 0$.

Find: For this problem:
a) Using the convolution integral solution method, determine the response $x(t)$ of the system. Clearly indicate the $t < T$ and $t > TT$ components of your solution.
b) If $T = \pi\sqrt{m/k}$, at what time does the first maximum in the response x(t) occur? Provide a graphical explanation/interpretation for your answer.



SOLUTION

For zero ICs:

$$x(t) = \frac{1}{m\omega_n}\int_0^t \sin\omega_n(t-\tau) f(\tau)\,d\tau \quad ; \quad \omega_n = \sqrt{k/m}$$

For $0 < t < T$:

$$x(t) = \frac{F_0}{m\omega_n T}\int_0^t \tau \sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega_n T}\left\{ \left.\frac{\tau\cos\omega_n(t-\tau)}{\omega_n}\right|_{\tau=0}^{\tau=t} - \frac{1}{\omega_n}\int_0^t \cos\omega_n(t-\tau)\,d\tau \right\}$$

$$= \frac{F_0}{m\omega_n T}\left\{ \left.\frac{\tau\cos\omega_n(t-\tau)}{\omega_n}\right|_{\tau=0}^{\tau=t} + \left.\frac{\sin\omega_n(t-\tau)}{\omega_n^2}\right|_{\tau=0}^{\tau=t} \right\}$$

$$= \frac{F_0}{m\omega_n T}\left\{ \frac{t}{\omega_n} - \frac{\sin\omega_n t}{\omega_n^2} \right\}$$

For $t > T$:

$$x(t) = \frac{1}{m\omega_n}\left\{\int_0^T f(\tau)\sin\omega_n(t-\tau)\,d\tau + \int_T^t f(\tau)\sin\omega_n(t-\tau)\,d\tau\right\}$$

$$= \frac{1}{m\omega_n}\left\{\int_0^T \frac{F_0\tau}{T}\sin\omega_n(t-\tau)\,d\tau + \int_T^t 0\sin\omega_n(t-\tau)\,d\tau\right\}$$

$$= \frac{F_0}{m\omega_n T}\int_0^T \tau\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega_n T}\left\{\left.\frac{\tau\cos\omega_n(t-\tau)}{\omega_n}\right|_{\tau=0}^{\tau=T} - \frac{1}{\omega_n}\int_0^T \cos\omega_n(t-\tau)\,d\tau\right\}$$

$$= \frac{F_0}{m\omega_n T}\left\{\left.\frac{\tau\cos\omega_n(t-\tau)}{\omega_n}\right|_{\tau=0}^{\tau=T} + \left.\frac{\sin\omega_n(t-\tau)}{\omega_n^2}\right|_{\tau=0}^{\tau=T}\right\}$$

$$= \frac{F_0}{m\omega_n T}\left\{\frac{T\cos\omega_n(t-T)}{\omega_n} + \frac{\sin\omega_n(t-T) - \sin\omega_n t}{\omega_n^2}\right\}$$