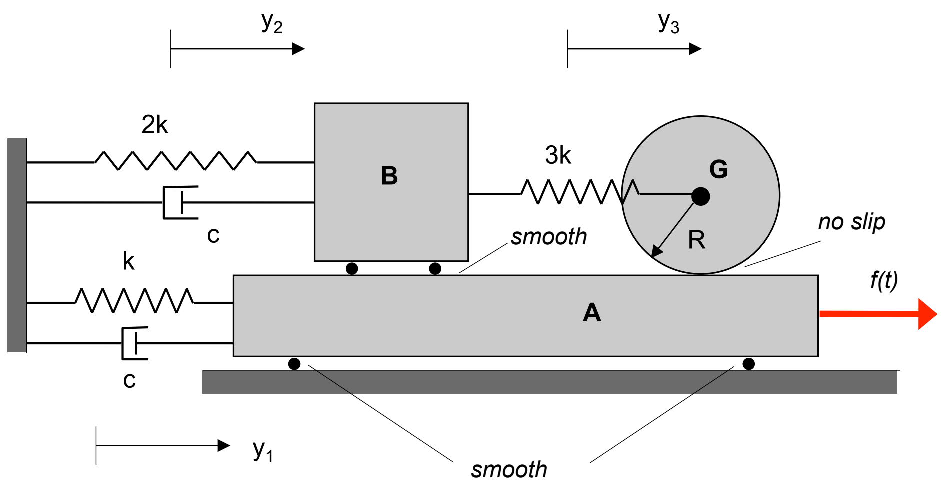
Example A4.15

Given: An external forcing is applied to the system below, where $f(t) = f_0 sin\Omega t$. The response of the system is to be described by the coordinates $y_1(t)$, $y_2(t)$ and $y_3(t)$. The particular solutions for these coordinates are to labeled as $y_{P1}(t)$, $y_{P2}(t)$ and $y_{P3}(t)$.

Find: For this problem:
a) At what values of the frequency Ω does resonance occur in the UNDAMPED system?
b) At what values (if any) of the frequency Ω do anti-resonances occur in the UNDAMPED system for $y_{P1}(t)$? For $y_{P2}(t)$? For $y_{P3}(t)$
c) Using the complex exponential approach, derive the form of the particular solutions $y_{P1}(t)$, $y_{P2}(t)$ and $y_{P3}(t)$ for the DAMPED system. Do not invert the matrix needed for solution.
d) Using Matlab, produce plots for the amplitudes of $y_{P1}(t)$, $y_{P2}(t)$ and $y_{P3}(t)$ vs. the excitation frequency Ω for four values of damping: $c/\sqrt{km} = 0, 0.2, 0.3, 0.4$.



SOLUTION

From Homework Problem No. 4.2:

$$[M]=\begin{bmatrix} \frac{3m}{2} & 0 & -\frac{m}{2} \\ 0 & m & 0 \\ -\frac{m}{2} & 0 & \frac{3m}{2} \end{bmatrix} \quad \text{and} \quad [K]=\begin{bmatrix} k & 0 & 0 \\ 0 & 5k & -3k \\ 0 & -3k & 3k \end{bmatrix}$$

Also,

$$dW = f dy_1 = Q_1 dy_1$$

Therefore, the undamped EOMs are:

$$[M]\ddot{\vec{y}}+[K]\vec{y}=\begin{Bmatrix} f(t) \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} sin\Omega t$$

The particular solution of the undamped EOMs is: $\vec{y}_P(t)=\vec{Y}e^{i\Omega t}$. Substituting this in the EOMs gives:

$$\vec{Y}=\begin{bmatrix} -\frac{3m}{2}\Omega^2+k & 0 & \frac{m}{2}\Omega^2 \\ 0 & -m\Omega^2+5k & -3k \\ \frac{m}{2}\Omega^2 & -3k & -\frac{3m}{2}\Omega^2+3k \end{bmatrix}^{-1}\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix}$$

$$=\frac{1}{\Delta}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{Bmatrix} f_0 \\ 0 \\ 2f_0 \end{Bmatrix}=\frac{1}{\Delta}\begin{Bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{Bmatrix} f_0$$

where:

$$A_{11}=\begin{vmatrix} -m\Omega^2+5k & -3k \\ -3k & -\frac{3m}{2}\Omega^2+3k \end{vmatrix}$$

$$=(-m\Omega^2+5k)(-\frac{3m}{2}\Omega^2+3k)-(3k)^2=\frac{3m^2}{2}\Omega^4-\frac{21mk}{2}\Omega^2+6k^2$$

$$A_{21}=\begin{vmatrix} 0 & -3k \\ -3k & -\frac{3m}{2}\Omega^2+3k \end{vmatrix}=-(3k)^2$$

$$A_{31}=\begin{vmatrix} 0 & \frac{m}{2}\Omega^2 \\ -m\Omega^2+5k & -3k \end{vmatrix}=-(-m\Omega^2+5k)(\frac{m}{2}\Omega^2)$$

Resonances occur when $\Omega=\omega_1\ OR\ \omega_2\ OR\ \omega_3$, where the three natural frequencies were found in Homework Problem No. 4.2 to be:

$$\omega_1=0.707\sqrt{\frac{k}{m}} \quad ; \quad \omega_2=0.954\sqrt{\frac{k}{m}} \quad ; \quad \omega_3=2.567\sqrt{\frac{k}{m}}$$

Anti-resonances for y_1 occur when:

$$A_{11} = 0 = \frac{3m^2}{2}\Omega^4 - \frac{21mk}{2}\Omega^2 + 6k^2 \quad \Rightarrow$$

$$\Omega^2 = \sqrt{\frac{1}{2}\left[\frac{21}{2} \pm \sqrt{\left(\frac{21}{2}\right)^2 - (4)\left(\frac{3}{2}\right)(6)}\right]\frac{k}{m}}$$

$$= 0.792\sqrt{\frac{k}{m}}, 2.52\sqrt{\frac{k}{m}}$$

Anti-resonances for y_2 do NOT occur since $A_{21} \neq 0$.

Anti-resonances for y_3 occur when:

$$A_{31} = 0 = -(-m\Omega^2 + 5k)(\frac{m}{2}\Omega^2) \quad \Rightarrow \quad \Omega = \sqrt{\frac{5k}{m}} = 2.24\ \Omega$$

Including damping, we have:

$$R = \frac{1}{2}c\dot{y}_1^2 + \frac{1}{2}c\dot{y}_2^2$$

from which we find:

$$[C] = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the damped EOMs of:

$$[M]\ddot{\vec{y}} + [C]\dot{\vec{y}} + [K]\vec{y} = \begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} sin\Omega t = \begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} Imag(e^{i\Omega t})$$

Assuming a solution of the form $\vec{y}_P(t) = \vec{Y}e^{i\Omega t}$ gives:

$$[-\Omega^2[M] + i\Omega[C] + [K]]\vec{Y} = \begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \vec{Y} = [-\Omega^2[M] + i\Omega[C] + [K]]^{-1}\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix}$$

On solution of the above, we have:

$$\vec{y}_P(t) = Imag(\vec{Y}e^{i\Omega t}) = Imag([-\Omega^2[M] + i\Omega[C] + [K]]^{-1}e^{i\Omega t})\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= Imag([-\Omega^2[M] + i\Omega[C] + [K]]^{-1})\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} cos\Omega t + Real([-\Omega^2[M] + i\Omega[C] + [K]]^{-1})\begin{Bmatrix} f_0 \\ 0 \\ 0 \end{Bmatrix} sin\Omega t$$

where:

$$-\Omega^2[M]+i\Omega[C]+[K]=\begin{bmatrix} -\frac{3m}{2}\Omega^2+k+ic\Omega & 0 & \frac{m}{2}\Omega^2 \\ 0 & -m\Omega^2+5k+ic\Omega & -3k \\ \frac{m}{2}\Omega^2 & -3k & -\frac{3m}{2}\Omega^2+3k \end{bmatrix}$$

$$=k\begin{bmatrix} 1-\frac{3}{2}\left(\sqrt{\frac{m}{k}}\Omega^2\right)^2+i\frac{c}{\sqrt{km}}\left(\sqrt{\frac{m}{k}}\Omega\right) & 0 & \frac{1}{2}\left(\sqrt{\frac{m}{k}}\Omega^2\right)^2 \\ 0 & 5-\left(\sqrt{\frac{m}{k}}\Omega^2\right)^2+i\frac{c}{\sqrt{km}}\left(\sqrt{\frac{m}{k}}\Omega\right) & -3 \\ \frac{1}{2}\left(\sqrt{\frac{m}{k}}\Omega^2\right)^2 & -3 & 3-\frac{3}{2}\left(\sqrt{\frac{m}{k}}\Omega^2\right)^2 \end{bmatrix}$$

Matlab code:

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clear

M=[3/2,0,-1/2;0,1,0;-1/2,0,3/2];
K=[1,0,0;0,5,-3;0,-3,3];

W=linspace(0,5,2000);nW=length(W);
zz=[0,0.2,0.3,0.4];nz=length(zz);

for jj=1:nz

   C=zz(jj)*[1,0,0;0,1,0;0,0,0];
   for ii=1:nW
     H=-W(ii)^2*M+K+i*W(ii)*C;
     Y(:,ii)=inv(H)*[1;0;0];
   end

   figure(1)
   plot(W,abs(Y(1,:)),'r'),hold on
   axis([0,3,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_1|/f_0')

   figure(2)
   plot(W,abs(Y(2,:)),'r'),hold on
   axis([0,3,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_2|/f_0')

   figure(3)
   plot(W,abs(Y(3,:)),'r'),hold on
   axis([0,3,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_3|/f_0')

end
figure(1), hold off,figure(2), hold off,figure(3), hold off
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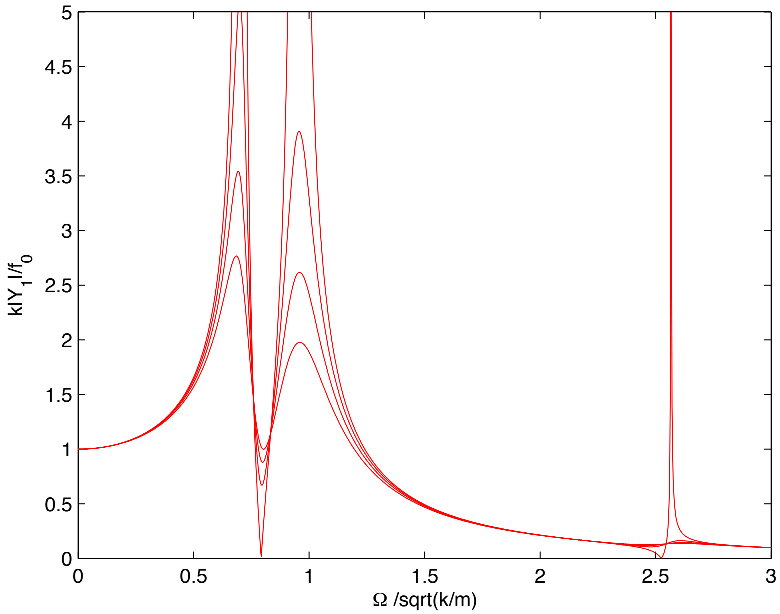
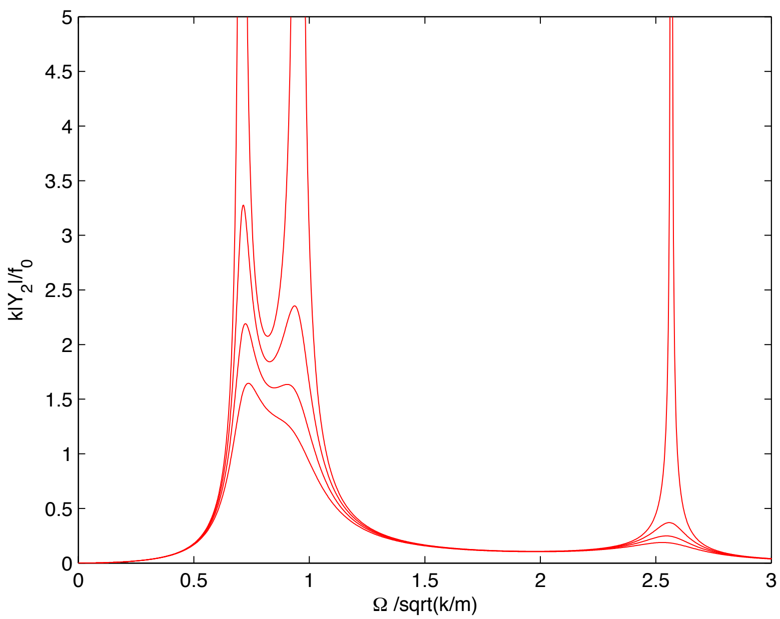
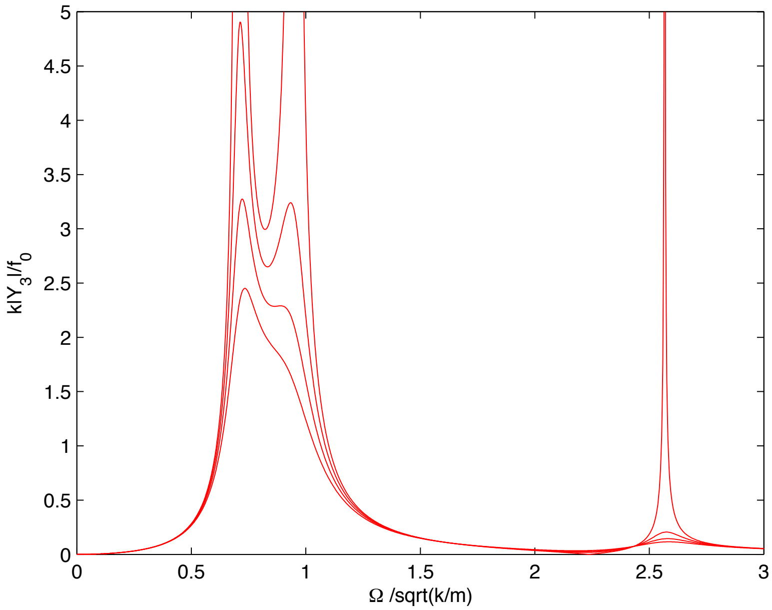

k|Y_1|/f_0
Ω /sqrt(k/m)
0 0.5 1 1.5 2 2.5 3
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5


k|Y_2|/f_0
Ω /sqrt(k/m)
0 0.5 1 1.5 2 2.5 3
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5



5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
$k|Y_3|/f_0$
Ω /sqrt(k/m)