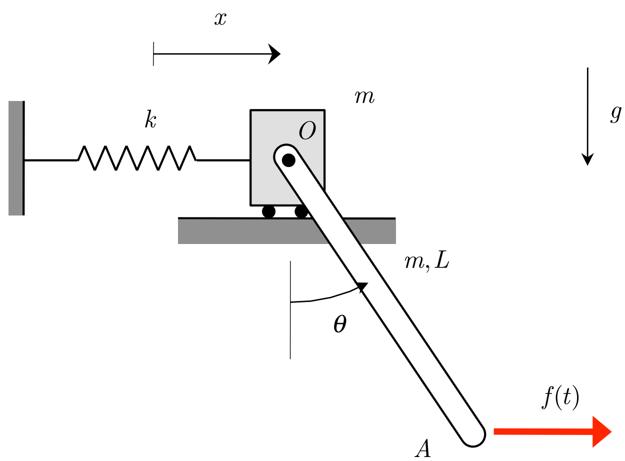
Example A4.14

Given: A forcing $f(t) = f_0 sin\Omega t$ acts at the end of the thin, homogeneous bar of the two-DOF system shown below. The response of the system is to be described by the coordinates $x(t)$ and $\theta(t)$. Let $g/L = 2k/m$.

Find: For this problem:
a) Derive the particular solutions $x_P(t)$ and $\theta_P(t)$ for the system.
b) At what values of the temporal frequency Ω does resonance occur in the system?
c) Show that the "shape" of the response is that of the first mode when excited at the first natural frequency, and that the shape is that of the second mode when excited at the second natural frequency.
d) At what values (if any) of the temporal frequency Ω do anti-resonances occur for $x_P(t)$? For $\theta_P(t)$
e) Make hand sketches for the amplitudes of $x_P(t)$ and $\theta_P(t)$ vs. the frequency Ω.



SOLUTION

From Homework Problem No. 4.1:

$$[M] = \begin{bmatrix} 2m & \dfrac{mL}{2} \\ \dfrac{mL}{2} & \dfrac{mL^2}{3} \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} k & 0 \\ 0 & \dfrac{mgL}{2} \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & kL^2 \end{bmatrix}$$

Also, we have:

$$\begin{aligned} dW &= (f\,\hat{i})\bullet d\vec{r}_A = (f\,\hat{i})\bullet d\left[(x + Lsin\theta)\hat{i} - Lcos\theta\,\hat{j}\right] \\ &= (f\,\hat{i})\bullet d\left[(dx + Lcos\theta\; d\theta)\hat{i} + Lsin\theta\; d\theta\,\hat{j}\right] \\ &= f\,dx + fLcos\theta\,d\theta \approx f\,dx + fL\,d\theta \\ &= Q_x\,dx + Q_\theta\,d\theta \end{aligned}$$

Therefore, the EOMs for the system are:

$$\begin{bmatrix} 2m & m/2 \\ m/2 & m/3 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ L\ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} = \begin{Bmatrix} f \\ f \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_0 sin\Omega t$$

Let $\vec{y}_P(t) = \begin{Bmatrix} x_P(t) \\ L\theta_P(t) \end{Bmatrix} = \vec{Y}\, sin\Omega t$. Substituting into the EOMs gives:

$$\begin{bmatrix} -2m\Omega^2 + k & -m\Omega^2/2 \\ -m\Omega^2/2 & -m\Omega^2/3 + k \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_0 \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} -2m\Omega^2 + k & -m\Omega^2/2 \\ -m\Omega^2/2 & -m\Omega^2/3 + k \end{bmatrix}^{-1} \begin{Bmatrix} f_0 \\ f_0 \end{Bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} -m\Omega^2/3 + k & m\Omega^2/2 \\ m\Omega^2/2 & -2m\Omega^2 + k \end{bmatrix} \begin{Bmatrix} f_0 \\ f_0 \end{Bmatrix}$$

$$= \frac{1}{\Delta} \begin{Bmatrix} -m\Omega^2/3 + k + m\Omega^2/2 \\ m\Omega^2/2 - 2m\Omega^2 + k \end{Bmatrix} f_0 = \frac{1}{\Delta} \begin{Bmatrix} m\Omega^2/6 + k \\ -3m\Omega^2/2 + k \end{Bmatrix} f_0$$

where (see Homework Problem No. 4.1 solution):

$$\Delta = m^2 \left[\left(\frac{5}{12}\right)\Omega^4 - \left(\frac{7k}{3m}\right)\Omega^2 + \left(\frac{k}{m}\right)^2 \right]$$

Resonances occur when $\Delta = 0$; that is, when $\Omega = \omega_1$ or $\Omega = \omega_2$, where the natural frequencies are given by (see results from Homework Problem No. 4.1:

$$\omega_1 = 0.684\sqrt{\frac{k}{m}}$$

$$\omega_2 = 2.27\sqrt{\frac{k}{m}}$$

At $\Omega = \omega_1$:

$$\frac{Y_2}{Y_1} = \frac{1 - 3m\omega_1^2/2k}{1 + m\omega_1^2/6k} = 0.277 \quad \text{(in \textit{first} mode, as expected)}$$

At $\Omega = \omega_2$:

$$\frac{Y_2}{Y_1} = \frac{1 - 3m\omega_2^2/2k}{1 + m\omega_2^2/6k} = -3.61 \quad \text{(in \textit{second} mode, as expected)}$$

There are no anti-resonances for x. However, when $\Omega = \sqrt{2k/3m} = 0.816\sqrt{k/m}$, θ has an anti-resonance.

For the hand sketch, you need to clearly identify the following:

- As $\Omega \to 0$, $|x| \to 2 f_0 / k$ and $L|\theta| \to f_0 / k$
- As $\Omega \to \infty$, $|x| \to 0^+$ and $L|\theta| \to 0^-$
- For $\Omega = \omega_1 \; or \; \omega_2$, $|x| \to \infty$ and $L|\theta| \to \infty$
- When $\Omega = \sqrt{2k/3m} = 0.816\sqrt{k/m}$, $|x| = 19 f_0 / 9k$ and $L|\theta| = 0$