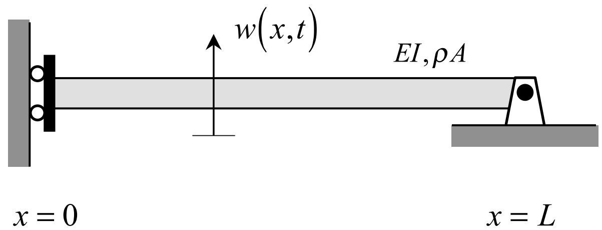
Example A3.4

Given: Consider the transverse motion of the thin beam shown below.

Find: For this problem:

a) Develop the characteristic equation (CE).
b) If a closed-from solution of the CE is possible, solve it. If not, then: i) make a hand sketch of the terms in the CE showing the locations of the roots of the CE; ii) place lower and upper bounds on the first four natural frequencies for the system; iii) using Matlab, or an equivalent application, determine numerical values for the first four natural frequencies, as well as the corresponding modal functions. Your answers should all be in terms of system parameters such as E, I, A, L and ρ.
c) Make a hand sketch of first four modal functions.



SOLUTION

EOM: $EI\dfrac{\partial^4 w}{\partial x^4}+\rho A\dfrac{\partial^2 w}{\partial t^2}=0$

Solution of EOM:

$$w(x,t)=W(x)T(t) \quad \Rightarrow$$

$$W(x)=acosh\beta x+bsinh\beta x+ccos\beta x+dsin\beta x \qquad ; \quad \beta^4=\frac{\rho A}{EI}\omega^2$$

Enforcement of BCs:

<u>At x = 0</u>:

$$\frac{\partial w}{\partial x}(0,t)=W'(0)T(t)=0 \quad \Rightarrow \quad W'(0)=0=\beta(b+d) \qquad (1)$$

$$EI\frac{\partial^3 w}{\partial x^3}(0,t)=W'''(0)T(t)=0 \quad \Rightarrow \quad W'''(0)=0=\beta^3(b-d) \qquad (2)$$

For $\beta\neq 0$, equations (1) and (2) show that: $b=d=0$

<u>At x = L</u>:

$$w(L,t)=W(L)T(t)=0 \quad \Rightarrow \quad W(L)=0=a\,cosh\beta L+c\,cos\beta L \qquad (3)$$

$$EI\frac{\partial^2 w}{\partial x^2}(0,t)=W''(0)T(t)=0 \quad \Rightarrow \quad W''(0)=0=\beta^2(a\,cosh\beta L-c\,cos\beta L) \qquad (4)$$

For $\beta \neq 0$, add together equations (3) and (4):

$$a\ cosh\beta L = 0 \quad \Rightarrow \quad a = 0 \quad (since\ cosh\beta L \neq 0)$$

Therefore, equation (3) gives:

$$c\ cos\beta L = 0 \tag{5}$$

Since $c = 0$ would correspond to a trivial solution (recall a = b = d = 0), we have:

$$cos\beta L = 0 \quad \Rightarrow \quad \beta_j L = \frac{2j-1}{2}\pi \quad \Rightarrow$$

$$\omega_j = (\beta_j L)^2 \sqrt{\frac{EI}{\rho A L^4}} = \left(\frac{2j-1}{2}\pi\right)^2 \sqrt{\frac{EI}{\rho A L^4}} \tag{6}$$

with modal functions of:

$$W^{(j)}(x) = c_j cos\beta_j x = cos\left(\frac{2j-1}{2}\pi\frac{x}{L}\right) \quad ; \quad c_j = 1$$

The free response of the beam is given by:

$$w(x,t) = \sum_{j=1}^{\infty} W^{(j)}(x)\left[C_j cos\omega_j t + S_j sin\omega_j t\right]$$

$$= \sum_{j=1}^{\infty} cos\left(\frac{2j-1}{2}\pi\frac{x}{L}\right)\left[C_j cos\left(\left(\frac{2j-1}{2}\pi\right)^2 \sqrt{\frac{EI}{\rho A L^4}}t\right) + S_j sin\left(\left(\frac{2j-1}{2}\pi\right)^2 \sqrt{\frac{EI}{\rho A L^4}}t\right)\right]$$

where the response coefficients C_j and S_j are found from the enforcement of the initial conditions.

Check:

We assumed above that $\beta \neq 0$. Is this valid? To check this, substitute $\beta = 0$ into the original equation for $W(x)$:

$$W(\beta = 0) = a + c = constant$$

Note that a constant function cannot have a zero value and a non-zero slope at x = L, as needed for the beam to move. Therefore, $\beta = 0$ will correspond to a trivial (no-motion) solution.