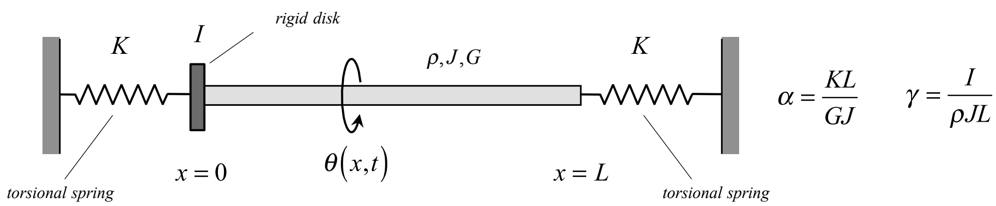
Example A3.3

Given: Consider the shaft shown below.

Find: For this problem:
a) Develop the characteristic equation (CE).
b) Make a hand sketch of the terms in the CE showing the locations of the roots of the CE.
c) Based on your sketch above, place lower and upper bounds on the first four natural frequencies for the system.
d) Using Matlab, or an equivalent application, determine numerical values for the first four natural frequencies, as well as the corresponding modal functions. Use $\alpha = 2$ and $\gamma = 1$ in your numerics. Your answers should be in terms of system parameters indicated in the figure.



EOM:

$$GJ\frac{\partial^2\theta}{\partial x^2} = \rho J\frac{\partial^2\theta}{\partial t^2}$$

Solution:

$$\theta(x,t) = \Theta(x)T(t) \quad \Rightarrow$$

$$\ddot{T}(t) = -\omega^2 T(t)$$

$$\Theta(x) = acos\beta x + bsin\beta x \quad ; \quad \beta^2 = \frac{\rho}{G}\omega^2$$

Enforce BCs:

At x = 0:

$$\sum M = -K\theta(0,t) + GJ\frac{\partial\theta}{\partial x}(0,t) = I\frac{\partial^2\theta}{\partial t^2}(0,t) \quad \Rightarrow$$

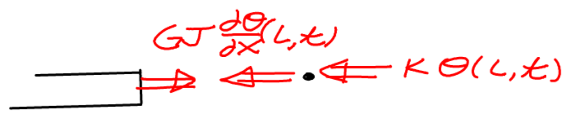
$$-K\Theta(0)T(t) + GJ\Theta'(0)T(t) = I\Theta(0)\ddot{T}(t) = -\omega^2 I\Theta(0)T(t) \quad \Rightarrow$$

$$GJ\Theta'(0) = (K - \omega^2 I)\Theta(0) \quad \Rightarrow$$

$$GJb\beta = (K - \omega^2 I)a \quad \Rightarrow$$

$$\frac{b}{a} = \frac{K - \omega^2 I}{GJ\beta} = \frac{K - \beta^2 IG/\rho}{GJ\beta} = \frac{KL}{GJ}\frac{1}{\beta L} - \frac{I}{\rho JL}\beta L = \frac{\alpha}{\beta L} - \gamma(\beta L) \qquad (1)$$

At x = L:



$$\sum M = -K\theta(L,t) - GJ\frac{\partial\theta}{\partial x}(L,t) = 0 \quad \Rightarrow$$

$$-K\Theta(L)T(t) - GJ\Theta'(L)T(t) = 0 \quad \Rightarrow$$

$$GJ\Theta'(L) = -K\Theta(L) \quad \Rightarrow$$

$$GJ\beta(-asin\beta L + bcos\beta L) = -K(acos\beta L + bsin\beta L) \quad \Rightarrow$$

$$-aGJ\beta tan\beta L + bGJ\beta = -aK - bKtan\beta L \quad \Rightarrow$$

$$-atan\beta L + b = -a\frac{KL}{GJ}\frac{1}{\beta L} - b\frac{KL}{GJ}\frac{1}{\beta L}tan\beta L = -a\frac{\alpha}{\beta L} - b\frac{\alpha}{\beta L}tan\beta L \quad \Rightarrow$$

$$\left(\frac{\alpha}{\beta L} - tan\beta L\right)a = -\left(1 + \frac{\alpha}{\beta L}tan\beta L\right)b \quad \Rightarrow$$

$$\frac{b}{a}=\frac{-\frac{\alpha}{\beta L}+tan\beta L}{1+\frac{\alpha}{\beta L}tan\beta L} \tag{2}$$

Equating (1) and (2):

$$\frac{\alpha}{\beta L}-\gamma(\beta L)=\frac{-\frac{\alpha}{\beta L}+tan\beta L}{1+\frac{\alpha}{\beta L}tan\beta L} \quad \Rightarrow$$

$$\left[\frac{\alpha}{\beta L}-\gamma(\beta L)\right]\left[1+\frac{\alpha}{\beta L}tan\beta L\right]=-\frac{\alpha}{\beta L}+tan\beta L \quad \Rightarrow$$

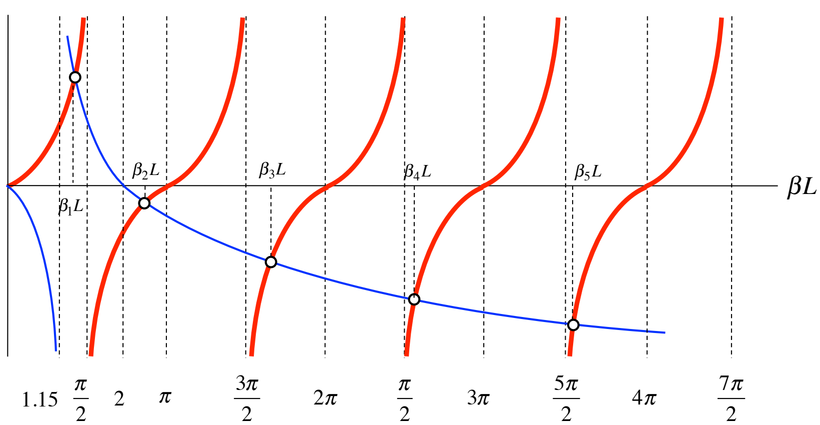
$$\left\{\left[\frac{\alpha}{\beta L}-\gamma(\beta L)\right]\frac{\alpha}{\beta L}-1\right\}tan\beta L=-2\frac{\alpha}{\beta L}+\gamma(\beta L) \quad \Rightarrow$$

$$tan\beta L=\frac{-2\frac{\alpha}{\beta L}+\gamma(\beta L)}{\left[\frac{\alpha}{\beta L}-\gamma(\beta L)\right]\frac{\alpha}{\beta L}-1}=\frac{\beta L\left[\gamma(\beta L)^2-2\alpha\right]}{-(\beta L)^2(1+\alpha\gamma)+\alpha^2}=\frac{g(\beta L)}{h(\beta L)}$$

Notes:

- $g(\beta L)=0$ when: $\beta L=0,\sqrt{2\alpha/\gamma}=0,2$
- $h(\beta L)=0$ when: $\beta L=\sqrt{\frac{\alpha^2}{1+\alpha\gamma}}=\frac{2}{\sqrt{3}}=1.15$.

Therefore, the RHS has zeros at $\beta L=0,2$ and a vertical asymptote at $\beta L=1.15$.



From figure above the bounds for the roots of the characteristic equation are:

$$1.15 < \beta_1 L < \frac{\pi}{2} \qquad \Rightarrow \qquad 1.15\sqrt{\frac{G}{\rho L^2}} < \omega_1 < \frac{\pi}{2}\sqrt{\frac{G}{\rho L^2}}$$

$$2 < \beta_2 L < \pi \qquad \Rightarrow \qquad 2\sqrt{\frac{G}{\rho L^2}} < \omega_2 < \pi\sqrt{\frac{G}{\rho L^2}}$$

$$\frac{3\pi}{2} < \beta_3 L < 2\pi \qquad \Rightarrow \qquad \frac{3\pi}{2}\sqrt{\frac{G}{\rho L^2}} < \omega_3 < 2\pi\sqrt{\frac{G}{\rho L^2}}$$

$$\frac{5\pi}{2} < \beta_4 L < 3\pi \qquad \Rightarrow \qquad \frac{5\pi}{2}\sqrt{\frac{G}{\rho L^2}} < \omega_4 < 3\pi\sqrt{\frac{G}{\rho L^2}}$$

Numerical values for the first four roots of the CE are found using the Matlab routine 'fsolve'. The natural frequencies and modal functions are found from:

$$\omega_j = \beta_j L\sqrt{\frac{G}{\rho L^2}}$$

$$\Theta^{(j)}(x) = cos\beta_j x + \left(\frac{b}{a}\right)^{(j)} sin\beta_j x$$

respectively. Numerical values from fsolve are found below:

initial guess	$\beta_j L$	$(b/a)^{(j)}$
1.2	1.279	0.2846
2.5	2.6863	-1.9418
5.0	5.2762	-4.8971
8.0	8.2175	-7.9741

Matlab code listing:

```
clear

x0=1.2;
x=fsolve('func06',x0)
ba=2/x-x

x0=2.5;
x=fsolve('func06',x0)
ba=2/x-x

x0=5.5;
x=fsolve('func06',x0)
ba=2/x-x

x0=8.5;
x=fsolve('func06',x0)
ba=2/x-x
```

where in a separate file named "func06.m' is:

```
function f=func06(x)
f=tan(x)-x*(x^2-4)/(-3*x^2+4);
```

Questions: Since we are using a numerical method to solve the characteristic equation (CE), why do we need to make a sketch of the functions in the CE? Furthermore, why do we need to find upper and lower bounds on the roots of the CE?