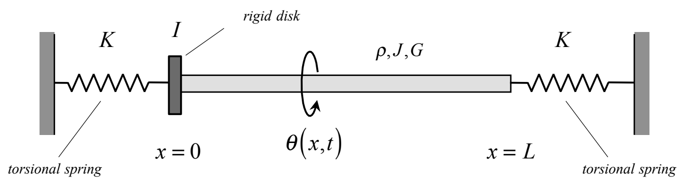
Example A3.2

Given: Consider the three continuous systems shown below.

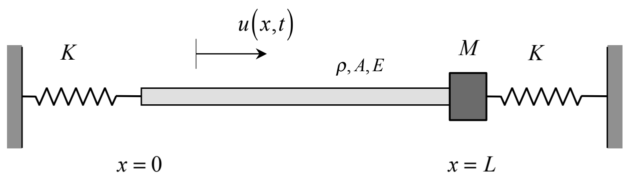
Find: For each system, DERIVE the boundary conditions at both $x = 0$ and $x = L$. Your derivations must include appropriate free body diagrams.

a) torsional motion of shaft



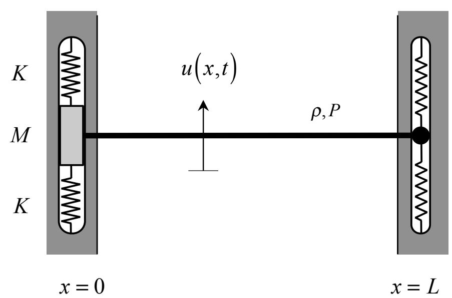
$$\alpha = \frac{KL}{GJ} \qquad \gamma = \frac{I}{\rho JL}$$

b) axial motion of rod



$$\alpha = \frac{KL}{EA} \qquad \gamma = \frac{M}{\rho AL}$$

c) transverse motion of string



$$\alpha = \frac{KL}{P} \qquad \gamma = \frac{M}{\rho L}$$

Problem 6.1b

EOM:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2}$$

Solution:

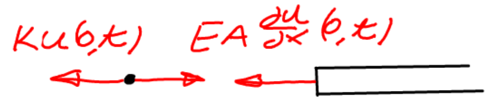
$$u(x,t) = U(x)T(t) \quad \Rightarrow$$

$$\ddot{T}(t) = -\omega^2 T(t)$$

$$\Theta(x) = a cos\beta x + b sin\beta x \quad ; \quad \beta^2 = \frac{\rho}{G}\omega^2$$

Enforce BCs:

At x = 0:

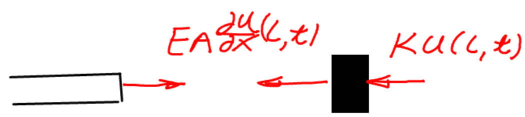


$$\sum F_x = -Ku(0,t) + EA\frac{\partial u}{\partial x}(0,t) = 0 \quad \Rightarrow$$

$$-KU(0)T(t) + GJU'(0)T(t) = 0 \quad \Rightarrow$$

$$GJU'(0) = KU(0) \tag{1}$$

At x = L:



$$\sum F_x = -Ku(L,t) - EA\frac{\partial u}{\partial x}(L,t) = M\frac{\partial^2 u}{\partial x^2}(L,t) \quad \Rightarrow$$

$$-KU(L)T(t) - EAU'(L)T(t) = MU(L)\ddot{T}(t) = -\omega^2 MU(L)T(t) \quad \Rightarrow$$

$$-EAU'(L) = (K - \omega^2 M)U(L) \tag{2}$$

Problem 6.1c

EOM:

$$P\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

Solution:

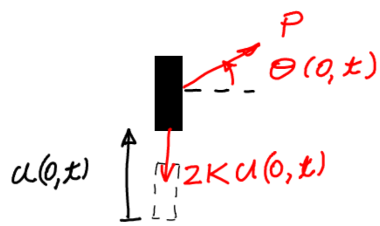
$$u(x,t) = U(x)T(t) \quad \Rightarrow$$

$$\ddot{T}(t) = -\omega^2 T(t)$$

$$U(x) = a cos\beta x + b sin\beta x \quad ; \quad \beta^2 = \frac{\rho}{G}\omega^2$$

Enforce BCs:

At x = 0:



$$\sum F_y = M\frac{\partial^2 m}{\partial t^2}(0,t) = -2Ku(0,t) + P sin\theta(0,t)$$

$$= -2Ku(0,t) + P\frac{\partial u}{\partial x}(0,t) \quad \Rightarrow$$

$$-2KU(0)T(t) + PU'(0)T(t) = MU(0)\ddot{T}(t) = -\omega^2 MU(0)T(t) \quad \Rightarrow$$

$$PU'(0) = (2K - \omega^2 M)U(0) \tag{1}$$

At x = L:

$$\sum F = -Ku(L,t) - P\frac{\partial u}{\partial x}(L,t) = 0 \quad \Rightarrow$$

$$-2KU(L)T(t) - PU'(L)T(t) = 0 \quad \Rightarrow$$

$$PU'(L) = -KU(L) \tag{2}$$