Example A3.1

Given: Consider the axial motion of the thin rod shown below. The rod is made up of a material having Young's modulus E and mass density ρ, and has a cross-sectional area of A. The rod has a free end at $x = 0$ and fixed end at $x = L$.

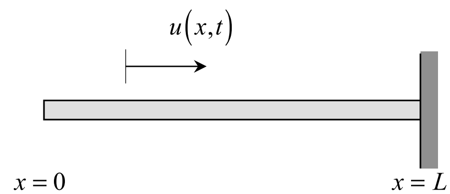
Find: For this problem:

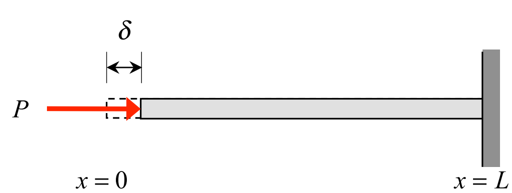
a) Determine the natural frequencies and modal functions for this rod, ω_j and $U^{(j)}(x)$, respectively, for $j = 1, 2, 3, \ldots$.

b) Suppose that an axial load P is applied to the left end of the rod in such a way that the rod is statically deformed with an end deformation of δ. This load is removed quickly with the rod still being at rest. If the subsequent motion of the rod is written as:

$$u(x,t) = \sum_{j=1}^{\infty} (c_j cos\omega_j t + s_j sin\omega_j t)\, U^{(j)}(x)$$

write down expressions for the response coefficients c_j and s_j.





EOM: $$EA \frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2}$$

Solution: $$u(x,t) = U(x)\,T(t)$$

↳ $$EAU''T = \rho A U \ddot{T}$$

↳ $$\underbrace{\frac{E}{\rho}\,\frac{U''(x)}{U(x)}}_{x} = \underbrace{\frac{\ddot{T}}{T}}_{t} \triangleq -\omega^2$$

$$\therefore \begin{cases} \ddot{T} + \omega^2 T = 0 \Rightarrow T(t) = C\cos\omega t + S\sin\omega t \\ U'' + \underbrace{\frac{\rho}{E}\omega^2}_{\hookrightarrow = \beta^2} U = 0 \Rightarrow U(x) = a\cos\beta x + b\sin\beta x \end{cases}$$

<u>Enforce BCs</u>

- $EA\frac{\partial u}{\partial x}(0,t) = EAU'(0)T(t) = 0 \Rightarrow$

  $U'(0) = 0 = -a\beta\sin\overset{0}{0} + b\beta\cos\overset{1}{0}$

  $\hookrightarrow b\beta = 0 \Rightarrow b = 0$ for $\beta \neq 0$

- $u(L,t) = U(L)T(t) = 0 \Rightarrow$

  $U(L) = 0 = a\cos\beta L$

  For $a \neq 0 \Rightarrow \cos\beta L = 0$

  $$\Rightarrow \beta L = \left(\frac{2j-1}{2}\right)\pi \; ; \; j = 1, 2, \ldots$$

  $$\Rightarrow \beta_j = \left(\frac{2j-1}{2}\right)\frac{\pi}{L} = \sqrt{\frac{\rho}{E}}\,\omega_j$$

  $$\hookrightarrow \omega_j = \left(\frac{2j-1}{2}\right)\pi\sqrt{\frac{E}{\rho L^2}}$$

  And, $U^{(j)}(x) = \overset{1}{a_j}\cos\frac{2j-1}{2}\pi\frac{x}{L}$

$$\therefore u(x,t) = \sum_{j=1}^{\infty} U^{(j)}(x)\left(C_j\cos\omega_j t + S_j\sin\omega_j t\right)$$

<u>Enforce ICs:</u>

- $u(x,0) = u_0(x) = \sum_{j=1}^{\infty} U^{(j)}(x)\,C_j$

  Multiply by $U^{(k)}(x)$ & $\int_0^L (\;)\,dx$:

  $$\int_0^L u_0(x)\,U^{(k)}(x)\,dx = \sum_{j=1}^{\infty}\underbrace{\left(\int_0^L U^{(k)}U^{(j)}\,dx\right)}_{=0\;;\;j\neq k} C_j$$

$$\therefore\ C_k = \frac{\int_0^L u_0(x)\, v^{(k)}(x)\, dx}{\int_0^L v^{(k)2}(x)\, dx}$$

- $$\frac{\partial u}{\partial t}(x,0) = 0 = \sum_j \omega_j S_j v^{(j)}(x)$$

  $\hookrightarrow S_j = 0\ ;\ j = 1, 2, \ldots$

<u>Note that:</u>

- $$\int_0^L v^{(k)2}(x)\, dx = \int_0^L \cos^2\left(\frac{2k-1}{2}\pi\frac{x}{L}\right) dx$$

  $$= \frac{1}{2}\int_0^L \left[1 + \cos\frac{2(2k-1)}{2}\pi\frac{x}{L}\right] dx$$

  $$= \frac{L}{2} + \frac{1}{2}\frac{L}{\pi(2k-1)}\left(\cancel{\sin(2k-1)\pi\frac{x}{L}}\right)_{x=0}^{x=L} \xrightarrow{} 0$$

  $$= \frac{L}{2}$$

- $$\int_0^L u_0(x)\, v^{(k)} dx = \delta\int_0^L \left(1 - \frac{x}{L}\right)\cos\beta_k x\, dx$$

  $$= \delta\left\{\frac{1}{\beta_k}\sin\beta_k x\Big|_{x=0}^{x=L} - \left[\frac{x}{L\beta_k}\sin\beta_k x\right]_{x=0}^{x=L} + \frac{1}{L\beta_k}\int_0^L \sin\beta_k x\, dx\right\}$$

  $$= \delta\left\{\cancel{\frac{\sin\beta_k L}{\beta_k}} - \cancel{\frac{\sin\beta_k L}{\beta_k}} - \frac{1}{L\beta_k^2}\cos\beta_k x\Big|_0^L\right\}$$

  $$= -\delta\left(\frac{\cancel{\cos\beta_k L} - 1}{L\beta_k^2}\right) \xrightarrow{} 0$$

  $$= \frac{\delta}{L\beta_k^2}$$

$$\therefore \quad C_k = \frac{2\delta}{(\beta_k L)^2} = 2\delta\left[\frac{2}{\pi(2k-1)}\right]^2$$

$\dot{\varepsilon}$

$$u(x,t) = \sum_k C_k U^{(k)}_{(x)} \cos \omega_k t$$

$$u(x,t) = \frac{8\delta}{\pi^2} \sum \frac{\cos\left(\frac{2k-1}{2}\right)\pi\frac{x}{L}}{(2k-1)^2} \cos\sqrt{\frac{E}{\rho L^2}}\left(\frac{2k-1}{2}\right)\pi t$$