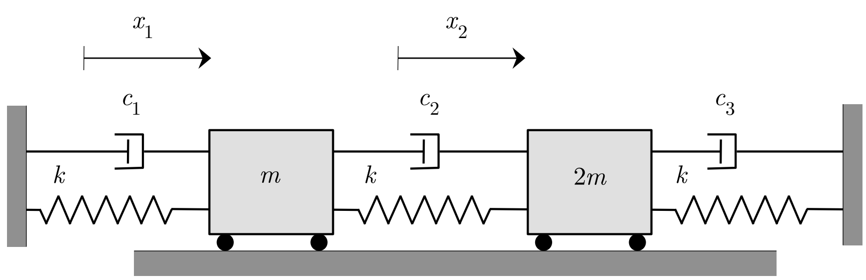
Example A2.21

Given: Consider the damped two-DOF system shown below whose motion is to be described by the absolute generalized coordinates x_1 and x_2.

Find: For this problem:
a) Determine the undamped natural frequencies and modal vectors for the system.
b) Suppose we would like to create a Rayleigh-damped system: $[C] = \alpha[M] + \beta[K]$ where $\alpha = c/m$ and $\beta = 2c/k$. Determine values for c_2, c_2 and c_3 that produces this desired Rayleigh damping. These values should be in terms of the parameter c.
c) Write down the two modally-uncoupled EOMs. What are the two modal damping ratios ζ_1 and ζ_2 corresponding to $c/\sqrt{km} = 0.1$?



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k x_2^2$$

$$R = \frac{1}{2} c_1 \dot{x}_1^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} c_3 \dot{x}_2^2$$

$\therefore$

$$[M] = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \; ; \; [C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

For undamped response:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0}$$

Solution: $\vec{x}(t) = \vec{X} e^{i\omega t}$

$\hookrightarrow$ $$\left[-\omega^2 [M] + [K]\right] \vec{X} = \vec{0}$$

For $\vec{X} \neq \vec{0} \Longrightarrow$

$$0 = \det\left[-\omega^2[M] + [K]\right]$$

$$= \begin{vmatrix} -m\omega^2+2k & -k \\ -k & -2m\omega^2+2k \end{vmatrix}$$

$$= (-m\omega^2+2k)(-2m\omega^2+2k) - k^2$$

$$= 2m^2\omega^4 - 6mk\omega^2 + 3k^2 \leftarrow \text{CE}$$

Solving CE:

$$\omega_{1,2}^2 = \frac{6 \pm \sqrt{6^2 - (4)(2)(3)}}{(2)(2)} \frac{k}{m}$$

$$= \frac{6 \pm \sqrt{12}}{4} \frac{k}{m} = \frac{3 \pm \sqrt{3}}{2} \frac{k}{m}$$

$$\hookrightarrow \begin{cases} \omega_1 = \sqrt{\dfrac{3-\sqrt{3}}{2}} \sqrt{\dfrac{k}{m}} \\ \omega_2 = \sqrt{\dfrac{3+\sqrt{3}}{2}} \sqrt{\dfrac{k}{m}} \end{cases}$$

<u>Modal vectors</u>

$$\begin{bmatrix} -m\omega^2+2k & -k \\ -k & -2m\omega^2+2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Using 1^{st} equation:

$$\left(\frac{X_2}{X_1}\right)^{(j)} = \frac{2k - m\omega_j^2}{k} = 2 - \frac{m}{k}\omega_j^2$$

For $X_1^{(j)} = 1$ for $j = 1, 2$, we have:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ \frac{1}{2}(1+\sqrt{3}) \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ \frac{1}{2}(1-\sqrt{3}) \end{Bmatrix}$$

Mass normalize modal vectors:

$$\vec{X}^{(1)T}[M]\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ \frac{1}{2}(1+\sqrt{3}) \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1 \\ \frac{1}{2}(1+\sqrt{3}) \end{Bmatrix}$$

$$= \left[1+\frac{1}{2}(1+\sqrt{3})^2\right]m$$

$$\vec{X}^{(2)T}[M]\vec{X}^{(2)} = \left[1+\frac{1}{2}(1-\sqrt{3})^2\right]m$$

$$\hat{\vec{X}}^{(j)} = \frac{1}{\sqrt{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}}\vec{X}^{(j)}$$

For Rayleigh damping:

$$[C] = \alpha[M] + \beta[K]$$

$$\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} = \alpha[M] + \beta[K]$$

$$= \frac{c}{m}\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} + \frac{2c}{k}\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$= \begin{bmatrix} 5c & -2c \\ -2c & 6c \end{bmatrix}$$

$$\left.\begin{aligned} &\underline{(1,1)}: \quad c_1+c_2 = 5c \\ &\underline{(1,2)}: \quad -c_2 = -2c \Rightarrow c_2 = 2c \end{aligned}\right\} \Rightarrow c_1 = 3c$$

$$(2,2): \quad c_2+c_3 = 6c \Rightarrow c_3 = 4c$$

<u>Modally uncoupled EOMS</u>

$$\vec{x}(t) = \sum_{j=1}^{2} \hat{\vec{X}}^{(j)} p_j(t)$$

$$\ddot{p}_j + 2\zeta_j\omega_j\dot{p}_j + \omega_j^2 p_j = 0$$

where:

$$2\zeta_j\omega_j = \hat{\vec{X}}^{(j)T}[C]\hat{\vec{X}}^{(j)}$$

$$= \alpha \underbrace{\hat{\vec{X}}^{(j)T}[M]\hat{\vec{X}}^{(j)}}_{1} + \beta \underbrace{\hat{\vec{X}}^{(j)T}[K]\hat{\vec{X}}^{(j)}}_{\omega_j^2}$$

$$= \frac{c}{m} + \frac{2c}{k}\omega_j^2$$

$$\zeta_j = \frac{1}{2}\left[\frac{c}{m\omega_j} + \frac{2c}{k}\omega_j\right]$$

$$\begin{cases} \zeta_1 = \frac{1}{2}\left[\sqrt{\frac{2}{3-\sqrt{3}}} + 2\sqrt{\frac{3-\sqrt{3}}{2}}\right]\frac{c}{\sqrt{km}} & \frac{c}{\sqrt{km}} = 0.1 \\ \zeta_2 = \frac{1}{2}\left[\sqrt{\frac{2}{3+\sqrt{3}}} + 2\sqrt{\frac{3+\sqrt{3}}{2}}\right]\frac{c}{\sqrt{km}} & \frac{c}{\sqrt{km}} = 0.1 \end{cases}$$