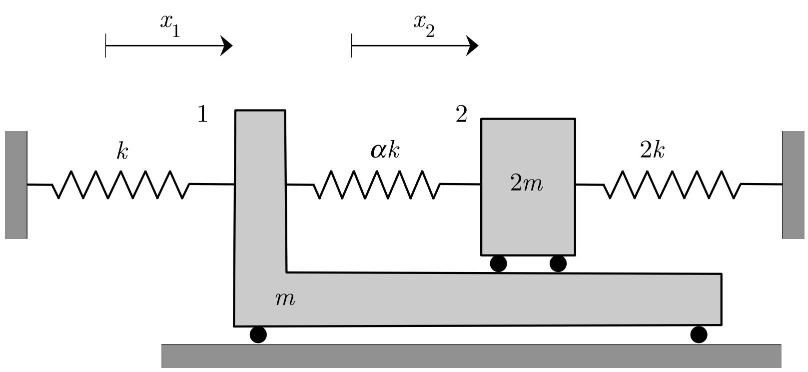
Example A2.20

Given: Consider the two-DOF system shown below whose motion is to be described by the absolute generalized coordinates x_1 and x_2.

Find: For this problem:
a) Determine the natural frequencies and modal vectors for the system.
b) Determine the beat period of response for the system corresponding to $\alpha << 1$.



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2}(\alpha k)(x_2 - x_1)^2 + \frac{1}{2}(2k) x_2^2$$

$$\therefore \quad [M] = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \; ; \; [K] = \begin{bmatrix} k+\alpha k & -\alpha k \\ -\alpha k & 2k+\alpha k \end{bmatrix}$$

<u>EOM</u>: $[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0}$

<u>SOLN</u>: $\vec{x}(t) = \vec{X} e^{i\omega t} \Rightarrow$

$$\left[-\omega^2 [M] + [K]\right] \vec{X} = \vec{0}$$

For $\vec{X} \neq \vec{0} \Rightarrow$

$$0 = \det\left[-\omega^2 [M] + [K]\right]$$

$$= \begin{bmatrix} -m\omega^2 + k + \alpha k & -\alpha k \\ -\alpha k & -2m\omega^2 + 2k + \alpha k \end{bmatrix}$$

$$= \left[-m\omega^2 + (1+\alpha)k\right]\left[-2m\omega^2 + (2+\alpha)k\right] - (\alpha k)^2$$

$$= 2m^2\omega^4 - (4+3\alpha)mk\omega^2 + \underbrace{\left[(1+\alpha)(2+\alpha) - \alpha^2\right]}_{2+3\alpha} k^2$$

$$\therefore \quad \omega_{1,2}^2 = \frac{(4+3\alpha) \pm \sqrt{(4+3\alpha)^2 - (4)(2)(2+3\alpha)}}{(2)(2)} \frac{k}{m}$$

$$= \left[\frac{(4+3\alpha) \pm 3\alpha}{4}\right] \frac{k}{m}$$

$$= \frac{k}{m}, \; \left(1 + \frac{3}{2}\alpha\right)\frac{k}{m}$$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{1 + \frac{3}{2}\alpha}\,\sqrt{\frac{k}{m}}$$

<u>Modal vectors</u>

$$\begin{bmatrix} -m\omega^2 + (1+\alpha)k & -\alpha k \\ -\alpha k & -2m\omega^2 + (2+\alpha)k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Using 1st equation:

$$\left(\frac{X_2}{X_1}\right)^{(j)} = \frac{(1+\alpha)k - m\omega_j^2}{\alpha k} = \frac{1 + \alpha - \frac{m}{k}\omega_j^2}{\alpha}$$

Let $X_1^{(j)} = 1$ for both modes. $\Rightarrow$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \; ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

For small α:

$$\omega_2 = \sqrt{1 + \frac{3}{2}\alpha}\,\sqrt{\frac{k}{m}}$$

$$\cong \left(1 + \frac{3}{4}\alpha + \ldots\right)\sqrt{\frac{k}{m}}$$

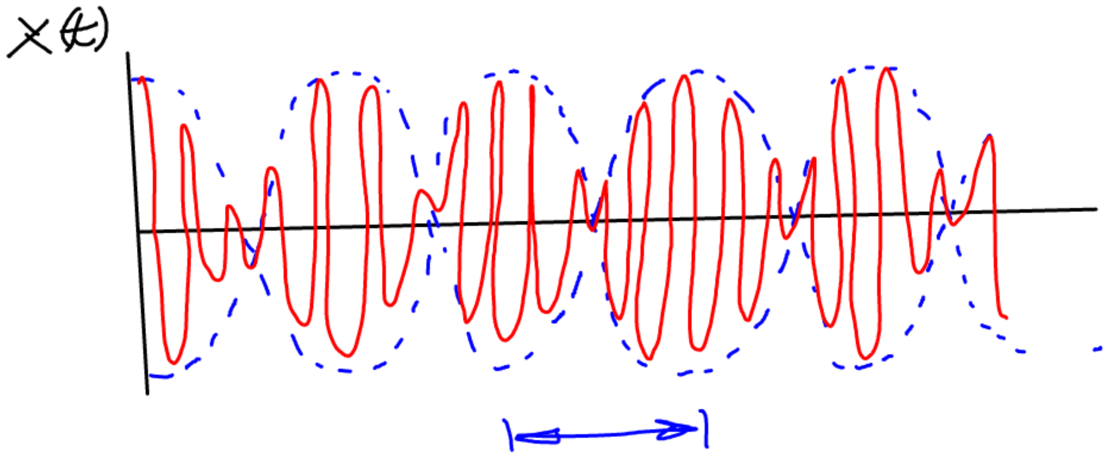
$$\therefore \quad \frac{\omega_2-\omega_1}{2} = \frac{\left(1+\frac{3}{4}\alpha\right)\sqrt{\frac{k}{m}} - \sqrt{\frac{k}{m}}}{2} = \frac{3}{8}\alpha\sqrt{\frac{k}{m}}$$

$$\frac{\omega_2+\omega_1}{2} \approx \omega_1 = \sqrt{\frac{k}{m}}$$

Response goes as:

$$x(t) \approx \cos\left(\frac{\omega_2-\omega_1}{2}\right)t \cos\left(\frac{\omega_1+\omega_2}{2}\right)t$$

$$= \cos\left(\frac{3}{2}\alpha\sqrt{\frac{k}{m}}\,t\right)\cos\sqrt{\frac{k}{m}}\,t$$



$$T_{beat} = \frac{\pi}{\frac{3}{2}\alpha\sqrt{\frac{k}{m}}} = \frac{2}{3}\frac{\pi}{\alpha}\sqrt{\frac{m}{k}}$$