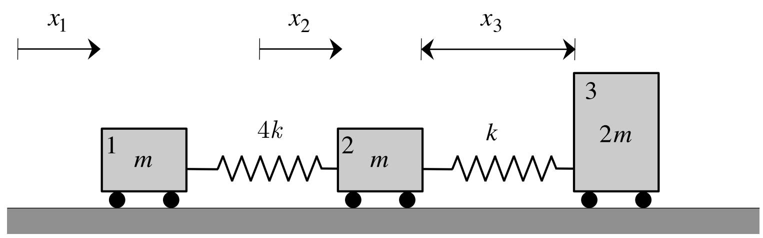
Example A2.19

Given: The system shown below is released from rest with the initial displacement conditions of $x_1(0) = x_2(0) = 0$ and $x_3(0) = A$.

Find: Determine the responses $x_1(t)$, $x_2(t)$ and $x_3(t)$.



SOLUTION

Energy expressions

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}(2m)(\dot{x}_2 + \dot{x}_3)^2$$
$$= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(3m)\dot{x}_2^2 + \frac{1}{2}(2m)\dot{x}_3^2 + \frac{1}{2}(4m)\dot{x}_2\dot{x}_3$$
$$= \frac{1}{2}M_{11}\dot{x}_1^2 + \frac{1}{2}M_{22}\dot{x}_2^2 + \frac{1}{2}M_{33}\dot{x}_3^2 + \frac{1}{2}(M_{23} + M_{32})\dot{x}_2\dot{x}_3$$

$$U = \frac{1}{2}(4k)(x_2 - x_1)^2 + \frac{1}{2}kx_3^2$$

Therefore:

$$[M] = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 2m \\ 0 & 2m & 2m \end{bmatrix}$$

$$[K] = \left[\frac{\partial^2 U}{\partial x_i \partial x_j} \right] = \begin{bmatrix} 4k & -4k & 0 \\ -4k & 4k & 0 \\ 0 & 0 & k \end{bmatrix}$$

Note that $det[K] = 0$; therefore, we expect at least one zero-frequency (rigid body) mode in the system.

Characteristic equation

$$0 = det[-\omega^2[M]+[K]]$$

$$= det\begin{bmatrix} -m\omega^2+4k & -4k & 0 \\ -4k & -3m\omega^2+4k & -2m\omega^2 \\ 0 & -2m\omega^2 & -2m\omega^2+k \end{bmatrix}$$

$$= (-m\omega^2+4k)[(-3m\omega^2+4k)(-2m\omega^2+k)-(-2m\omega^2)^2] - (-4k)(-4k)(-2m\omega^2+k)$$

$$= (-m\omega^2+4k)[2m^2\omega^4 - 11mk\omega^2 + 4k^2] + 32mk^2\omega^2 - 16k^3$$

$$= -2m^3\omega^6 + 19m^2k\omega^4 - 16mk^2\omega^2$$

$$= (2m^2\omega^4 - 19mk\omega^2 + 16k^2)m\omega^2$$

Check

$det[M] = 2m^3$ and $det[K] = 0$, which are in agreement with the first and last coefficients, respectively, in the above CE.

Natural frequencies

Solving the characteristic equation for the three roots of ω:

$$\omega_1 = 0$$

$$\omega_{2,3}^2 = \frac{19 \pm \sqrt{19^2-(4)(2)(16)}}{(2)(2)}\frac{k}{m} = \frac{19 \pm \sqrt{233}}{4}\frac{k}{m} = 0.9339\frac{k}{m}, 8.566\frac{k}{m}$$

or,

$$\omega_2 = \sqrt{0.1691}\sqrt{\frac{k}{m}} = 0.966\sqrt{\frac{k}{m}}$$

$$\omega_3 = \sqrt{47.31}\sqrt{\frac{k}{m}} = 2.93\sqrt{\frac{k}{m}}$$

Modal vectors

$$[-\omega^2[M]+[K]]\vec{X} = \begin{bmatrix} -m\omega^2+4k & -4k & 0 \\ -4k & -3m\omega^2+4k & -2m\omega^2 \\ 0 & -2m\omega^2 & -2m\omega^2+k \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using the first and third equation:

$$\left(\frac{X_2}{X_1}\right)^{(j)} = \frac{4k - m\omega_j^2}{4k} = \frac{4-(m/k)\omega_j^2}{4}$$

$$\left(\frac{X_3}{X_2}\right)^{(j)} = \frac{2(m/k)\omega_j^2}{1-2(m/k)\omega_j^2} \quad \Rightarrow$$

$$\left(\frac{X_3}{X_1}\right)^{(j)} = \left(\frac{X_3}{X_2}\right)^{(j)}\left(\frac{X_2}{X_1}\right)^{(j)} = \left(\frac{2(m/k)\omega_j^2}{1-2(m/k)\omega_j^2}\right)\left(\frac{4-(m/k)\omega_j^2}{4}\right)$$

Using $X_1^{(j)} = 1$ for j = 1, 2, 3, we have the following:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.7665 \\ -1.6498 \end{Bmatrix} \quad ; \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.1415 \\ 1.2123 \end{Bmatrix}$$

<u>Mass normalizing the modes</u>:

$$\alpha_1 = \frac{1}{\sqrt{\vec{X}^{(1)T}[M]\vec{X}^{(1)}}} = \frac{0.5}{\sqrt{m}}$$

$$\alpha_2 = \frac{1}{\sqrt{\vec{X}^{(2)T}[M]\vec{X}^{(2)}}} = \frac{0.5636}{\sqrt{m}}$$

$$\alpha_3 = \frac{1}{\sqrt{\vec{X}^{(3)T}[M]\vec{X}^{(3)}}} = \frac{0.6575}{\sqrt{m}}$$

from which we can write:

$$\hat{\vec{X}}^{(1)} = \begin{Bmatrix} 0.5 \\ 0.5 \\ 0 \end{Bmatrix}\frac{1}{\sqrt{m}} \quad ; \quad \hat{\vec{X}}^{(2)} = \begin{Bmatrix} 0.5636 \\ 0.4320 \\ -0.9299 \end{Bmatrix}\frac{1}{\sqrt{m}} \quad ; \quad \hat{\vec{X}}^{(3)} = \begin{Bmatrix} 0.6575 \\ -0.7506 \\ 0.7971 \end{Bmatrix}\frac{1}{\sqrt{m}}$$

<u>Response</u>

$$\vec{x}(t) = \sum_{j=1}^{2} \hat{\vec{X}}^{(j)}[c_j cos\omega_j t + s_j sin\omega_j t]$$

Enforcing ICs:

$$c_j = \hat{\vec{X}}^{(j)T}[M]\vec{x}(0) = \hat{\vec{X}}^{(j)T}\begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 2m \\ 0 & 2m & 2m \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ A \end{Bmatrix}$$

$$= 2mA\hat{\vec{X}}^{(j)T}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = 2mA(\hat{X}_2^{(j)} + \hat{X}_3^{(j)}) \quad \Rightarrow$$

$$c_1 = A\sqrt{m} \quad ; \quad c_2 = -0.9957A\sqrt{m} \quad ; \quad c_3 = 0.0931A\sqrt{m}$$

$$s_1 = \hat{\vec{X}}^{(1)}[M]\dot{\vec{x}}(0) = 0$$

$$s_j = \frac{1}{\omega_j}\hat{\vec{X}}^{(j)}[M]\dot{\vec{x}}(0) = 0 \quad ; \quad j = 2,3$$

Therefore,

$$\vec{x}(t) = \hat{\vec{X}}^{(1)} c_1 cos\omega_1 t + \hat{\vec{X}}^{(2)} c_2 cos\omega_2 t + \hat{\vec{X}}^{(3)} c_3 cos\omega_3 t$$

$$= A\begin{Bmatrix} 0.5 \\ 0.5 \\ 0 \end{Bmatrix} - 0.9957A\begin{Bmatrix} 0.5636 \\ 0.4320 \\ -0.9299 \end{Bmatrix} cos\left(0.966\sqrt{\frac{k}{m}}t\right)$$

$$+ 0.0931A\begin{Bmatrix} 0.6575 \\ -0.7506 \\ 0.7971 \end{Bmatrix} cos\left(2.93\sqrt{\frac{k}{m}}t\right)$$

The first term is non-oscillatory and represents the contribution of the rigid body mode to the response.