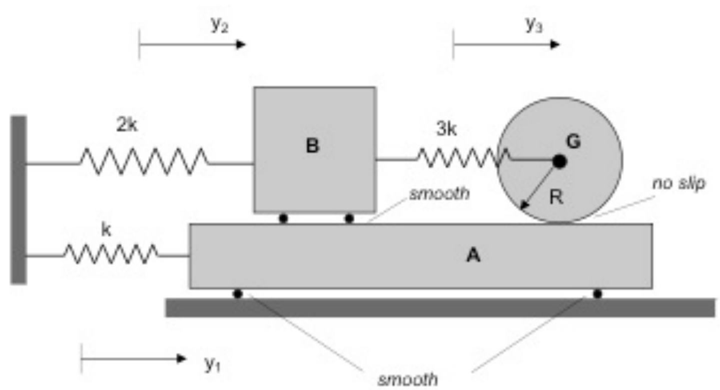
Example A2.18

Given: The absolute coordinates y_1, y_2 and y_3 are used to describe the motion of A, B and the center of mass G of the homogeneous wheel. Blocks A and B, as well as the wheel, each have a mass of m.

Find: For this problem:
a) Determine the mass and stiffness matrices for the system corresponding to the coordinates y_1, y_2 and y_3.
b) Derive the characteristic equation for the system. Express this characteristic equation in terms of non-dimensional natural frequencies $\omega/\sqrt{k/m}$.
c) Determine the natural frequencies from the characteristic equation found in b). You will need to use a numerical solver from Matlab (or Mathematica). Leave your final answers in terms of m and k.
d) Using your results from c), determine the modal vectors.
e) Numerically verify the orthogonality properties of the modal vectors: $\vec{Y}^{(i)T}[M]\vec{Y}^{(j)} = \vec{Y}^{(i)T}[K]\vec{Y}^{(j)} = 0;\ i \neq j$.



SOLUTION

Energy expressions

$$T = \frac{1}{2} m\dot{y}_1^2 + \frac{1}{2} m\dot{y}_2^2 + \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \dot{\theta}^2 \quad ; \quad I_G = \frac{1}{2} mR^2$$

$$U = \frac{1}{2} k y_1^2 + \frac{1}{2}(2k) y_2^2 + \frac{1}{2}(3k)(y_3 - y_2)^2$$

Kinematics

$$\dot{\theta} = \frac{\dot{y}_3 - \dot{y}_1}{R}$$

Therefore, the above KE expression become:

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}m\dot{y}_3^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{\dot{y}_3 - \dot{y}_1}{R}\right)^2 \\
&= \frac{1}{2}\left(\frac{3m}{2}\right)\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}\left(\frac{3m}{2}\right)\dot{y}_3^2 + \frac{1}{2}(-mR^2)\dot{y}_1\dot{y}_3 \\
&= \frac{1}{2}M_{11}\dot{y}_1^2 + \frac{1}{2}M_{22}\dot{y}_2^2 + \frac{1}{2}M_{33}\dot{y}_3^2 + \frac{1}{2}(M_{13} + M_{31})\dot{y}_1\dot{y}_3
\end{aligned}
$$

And, from the above, we have:

$$
[M] = \begin{bmatrix} M_{11} & 0 & M_{13} \\ 0 & M_{22} & 0 \\ M_{31} & 0 & M_{33} \end{bmatrix} = \begin{bmatrix} \frac{3m}{2} & 0 & -\frac{m}{2} \\ 0 & m & 0 \\ -\frac{m}{2} & 0 & \frac{3m}{2} \end{bmatrix} \quad \text{and} \quad [K] = \left[\frac{\partial^2 U}{\partial y_i \partial y_j}\right] = \begin{bmatrix} k & 0 & 0 \\ 0 & 5k & -3k \\ 0 & -3k & 3k \end{bmatrix}
$$

Eigenvalue problem

$$
\begin{aligned}
0 &= det[-\omega^2[M] + [K]] \\
&= det\begin{bmatrix} -\frac{3m}{2}\omega^2 + k & 0 & \frac{m}{2}\omega^2 \\ 0 & -m\omega^2 + 5k & -3k \\ \frac{m}{2}\omega^2 & -3k & -\frac{3m}{2}\omega^2 + 3k \end{bmatrix} \\
&= \left(-\frac{3}{2}m\omega^2 + k\right)\left[\left(-m\omega^2 + 5k\right)\left(-\frac{3m}{2}\omega^2 + 3k\right) - (3k)^2\right] \\
&\qquad + \left(\frac{m}{2}\omega^2\right)\left[-\left(\frac{m}{2}\omega^2\right)\left(-m\omega^2 + 5k\right)\right] \\
&= \left(-\frac{3}{2}m\omega^2 + k\right)\left[\frac{3}{2}m^2\omega^4 - \frac{21}{2}mk\omega^2 + 6k^2\right] + \frac{1}{4}m^3\omega^6 - \frac{5}{4}m^2k\omega^4 \\
&= -2m^3\omega^6 + 16m^2k\omega^4 - \frac{39}{2}mk^2\omega^2 + 6k^3
\end{aligned}
$$

Check

$det[M] = 2m^3$ and $det[K] = 6k^3$, which are in agreement with the first and last coefficients, respectively, in the above CE.

Natural frequencies

Divide above CE by k^3 and define $\mu = \frac{m}{k}\omega^2$:

$$0 = -2\frac{m^3}{k^3}\omega^6 + 16\frac{m^2}{k^2}\omega^4 - \frac{39}{2}\frac{m}{k}\omega^2 + 6$$
$$= -2\left(\frac{m}{k}\omega^2\right)^3 + 16\left(\frac{m}{k}\omega^2\right)^2 - \frac{39}{2}\left(\frac{m}{k}\omega^2\right) + 6$$
$$= -2\mu^3 + 16\mu^2 - \frac{39}{2}\mu + 6$$

Using Matlab function "roots" to solve the above CE:

$$\mu_1 = 0.5 \quad \Rightarrow \quad \omega_1 = \sqrt{\mu_1}\sqrt{\frac{k}{m}} = 0.707\sqrt{\frac{k}{m}}$$
$$\mu_2 = 0.9105 \quad \Rightarrow \quad \omega_2 = \sqrt{\mu_2}\sqrt{\frac{k}{m}} = 0.954\sqrt{\frac{k}{m}}$$
$$\mu_3 = 6.5895 \quad \Rightarrow \quad \omega_3 = \sqrt{\mu_3}\sqrt{\frac{k}{m}} = 2.567\sqrt{\frac{k}{m}}$$

Modal vectors

$$[-\omega^2[M] + [K]]\vec{X} = \vec{0} \quad \Rightarrow$$
$$\begin{bmatrix} -\frac{3m}{2}\omega^2 + k & 0 & \frac{m}{2}\omega^2 \\ 0 & -m\omega^2 + 5k & -3k \\ \frac{m}{2}\omega^2 & -3k & -\frac{3m}{2}\omega^2 + 3k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

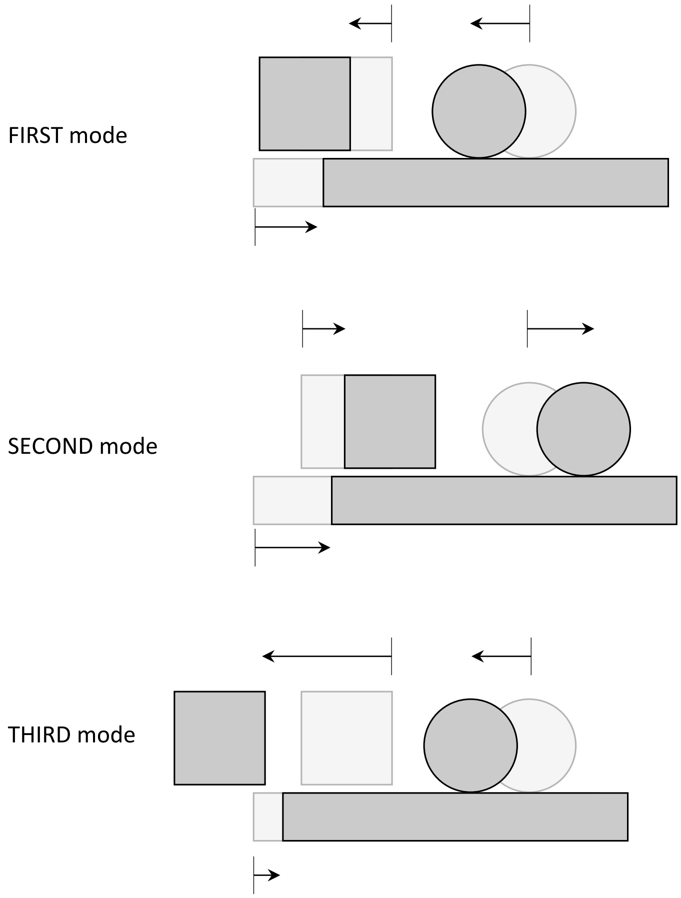
Using the first and second equations:

$$\left(\frac{X_3}{X_1}\right)^{(j)} = \frac{\frac{3m}{2}\omega_j^2 - k}{\frac{m}{2}\omega_j^2} = \frac{3\mu_j - 2}{\mu_j}$$
$$\left(\frac{X_2}{X_3}\right)^{(j)} = \frac{3k}{5k - m\omega^2} = \frac{3}{5 - \mu_j} \quad \Rightarrow$$
$$\left(\frac{X_2}{X_1}\right)^{(j)} = \left(\frac{X_2}{X_3}\right)^{(j)}\left(\frac{X_3}{X_1}\right)^{(j)} = \left(\frac{3}{5 - \mu_j}\right)\left(\frac{3\mu_j - 2}{\mu_j}\right)$$

Using $X_1^{(j)} = 1$ for j = 1, 2, 3 gives the following modal vectors:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -0.667 \\ -1 \end{Bmatrix} \quad ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.5895 \\ 0.8035 \end{Bmatrix} \quad ; \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -5.0895 \\ 2.6965 \end{Bmatrix}$$

The shapes of the three modal vectors are shown below:



Orthogonality is verified through the products:

$$\vec{X}^{(1)T}[M]\vec{X}^{(2)} = \vec{X}^{(1)T}[M]\vec{X}^{(3)} = \vec{X}^{(2)T}[M]\vec{X}^{(3)} = 0$$

$$\vec{X}^{(1)T}[K]\vec{X}^{(2)} = \vec{X}^{(1)T}[K]\vec{X}^{(3)} = \vec{X}^{(2)T}[K]\vec{X}^{(3)} = 0$$