Example A2.17

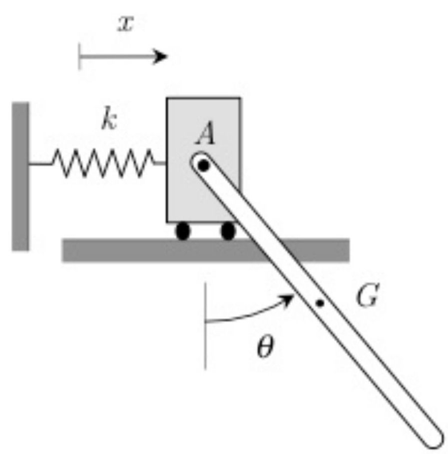
Given: The two-DOF system shown is described by the coordinates x and θ. The block and bar each have a mass of m. The thin bar is homogeneous in its mass distribution and has a length of L. Let $g/L = 2k/m$.

Find: For this problem:

a) Determine the mass and stiffness matrices for the linearized equations of motion for the system corresponding to small motion of the coordinates x and θ.

b) Determine the natural frequencies and modal vectors for the system. Leave your answers for frequencies in terms of m and k and for modal vectors in terms of L.

c) Determine the response of the system for initial conditions of $x(0) = A$, and $\theta(0) = \dot{x}(0) = \dot{\theta}(0) = 0$.



SOLUTION

Energy expressions

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \dot{\theta}^2 \quad ; \quad I_G = \frac{1}{12} mL^2$$

$$U = \frac{1}{2} kx^2 - mg\frac{L}{2} cos\theta$$

Kinematics

$$\vec{v}_G = \vec{v}_A + \vec{\omega} \times \vec{r}_{G/A}$$

$$= \dot{x}\hat{i} + (\dot{\theta}\hat{k}) \times \frac{L}{2}(sin\theta\,\hat{i} - cos\theta\,\hat{j})$$

$$= \left[\dot{x} + \left(\frac{L}{2} cos\theta \right) \dot{\theta} \right] \hat{i} + \left[\left(\frac{L}{2} sin\theta \right) \dot{\theta} \right] \hat{j} \quad \Rightarrow$$

$$v_G^2 = \left[\dot{x} + \left(\frac{L}{2}cos\theta\right)\dot{\theta}\right]^2 + \left[\left(\frac{L}{2}sin\theta\right)\dot{\theta}\right]^2$$
$$= \dot{x}^2 + \frac{L^2}{4}(cos^2\theta + sin^2\theta)\dot{\theta}^2 + (Lcos\theta)\dot{x}\dot{\theta}$$
$$= \dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2 + (Lcos\theta)\dot{x}\dot{\theta}$$

Therefore,

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2 + \frac{1}{4}mL^2\right)\dot{\theta}^2 + \frac{1}{2}(mLcos\theta)\dot{x}\dot{\theta}$$
$$= \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 + \frac{1}{2}(mLcos\theta)\dot{x}\dot{\theta}$$

From this, we have:

$$[M] = [m]_{\vec{q}_0} = \begin{bmatrix} 2m & \dfrac{mLcos\theta}{2} \\ \dfrac{mLcos\theta}{2} & \dfrac{mL^2}{3} \end{bmatrix}_{\vec{0}} = \begin{bmatrix} 2m & \dfrac{mL}{2} \\ \dfrac{mL}{2} & \dfrac{mL^2}{3} \end{bmatrix}$$

$$[K] = \left[\frac{\partial^2 U}{\partial q_i \partial q_j}\right]_{\vec{q}_0} = \begin{bmatrix} k & 0 \\ 0 & \dfrac{mgL}{2}cos\theta \end{bmatrix}_{\vec{0}} = \begin{bmatrix} k & 0 \\ 0 & \dfrac{mgL}{2} \end{bmatrix}$$

Eigenvalue problem

$$[-\omega^2[M] + [K]]\vec{X} = \vec{0} \quad \Rightarrow$$

$$\begin{bmatrix} -2m\omega^2 + k & -\dfrac{mL}{2}\omega^2 \\ -\dfrac{mL}{2}\omega^2 & -\dfrac{mL^2}{3}\omega^2 + \dfrac{mgL}{2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Characteristic equation

$$0 = det[-\omega^2[M] + [K]]$$
$$= (-2m\omega^2 + k)\left(-\frac{mL^2}{3}\omega^2 + \frac{mgL}{2}\right) - \left(\frac{mL}{2}\omega^2\right)^2$$
$$= \left(\frac{2m^2L^2}{3} - \frac{m^2L^2}{4}\right)\omega^4 - \left(\frac{mkL^2}{3} + m^2gL\right)\omega^2 + \frac{mgkL}{2}$$
$$= m^2L^2\left[\left(\frac{5}{12}\right)\omega^4 - \left(\frac{k}{3m} + \frac{g}{L}\right)\omega^2 + \left(\frac{1}{2}\frac{k}{m}\frac{g}{L}\right)\right] \quad ; \quad \frac{g}{L} = 2\frac{k}{m}$$
$$= m^2L^2\left[\left(\frac{5}{12}\right)\omega^4 - \left(\frac{7k}{3m}\right)\omega^2 + \left(\frac{k}{m}\right)^2\right]$$

Solving the CE:

$$\omega^2 = \left[\frac{(7/3) \pm \sqrt{(7/3)^2 - 4(5/12)}}{2(5/12)} \right] \frac{k}{m} = \frac{2}{5}(7 \pm \sqrt{34})\,\frac{k}{m} = 0.468\frac{k}{m},\ 5.13\frac{k}{m} \Rightarrow$$

$$\omega_1 = \sqrt{\frac{2}{5}(7 - \sqrt{34})}\,\sqrt{\frac{k}{m}} = 0.684\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{2}{5}(7 + \sqrt{34})}\,\sqrt{\frac{k}{m}} = 2.27\sqrt{\frac{k}{m}}$$

<u>*Modal vectors*</u>

Using the first equation from the eigenvalue problem:

$$\left(\frac{X_2}{X_1}\right)^{(j)} = \frac{-2m\omega_j^2 + k}{mL\omega_j^2/2} = \frac{2}{L}\left[\frac{1 - 2\omega_j^2(m/k)}{\omega_j^2(m/k)}\right]$$

Therefore:

$$\left(\frac{X_2}{X_1}\right)^{(1)} = \frac{2}{L}\left[\frac{1 - 2(0.468)}{(0.468)}\right] = \frac{0.277}{L}$$

$$\left(\frac{X_2}{X_1}\right)^{(1)} = \frac{2}{L}\left[\frac{1 - 2(5.13)}{(5.13)}\right] = -\frac{3.61}{L}$$

Using $X_1^{(j)} = 1$, the modal vectors become:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 0.277/L \end{Bmatrix} \quad ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -3.61/L \end{Bmatrix}$$

To mass-normalize the modal vectors:

$$\alpha_1 = \frac{1}{\sqrt{\vec{X}^{(1)T}[M]\vec{X}^{(1)}}} = \frac{1}{\sqrt{\begin{Bmatrix} 1 \\ 0.277/L \end{Bmatrix}^T \begin{bmatrix} 2m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.277/L \end{Bmatrix}}} = \frac{0.659}{\sqrt{m}}$$

$$\alpha_2 = \frac{1}{\sqrt{\vec{X}^{(2)T}[M]\vec{X}^{(2)}}} = \frac{1}{\sqrt{\begin{Bmatrix} 1 \\ -3.61/L \end{Bmatrix}^T \begin{bmatrix} 2m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix} \begin{Bmatrix} 1 \\ -3.61/L \end{Bmatrix}}} = \frac{0.605}{\sqrt{m}}$$

With this, we can write the mass-normalized modal vectors as:

$$\hat{\vec{X}}^{(1)} = \alpha_1 \vec{X}^{(1)} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.659 \\ 0.183/L \end{Bmatrix} \quad ; \quad \hat{\vec{X}}^{(2)} = \alpha_2 \vec{X}^{(2)} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.605 \\ -2.18/L \end{Bmatrix}$$

The free response is given by:

$$\vec{x}(t) = \sum_{j=1}^{2} \hat{\vec{X}}^{(j)}[c_j cos\omega_j t + s_j sin\omega_j t]$$

Enforcing the initial conditions:

$$c_1 = \hat{\vec{X}}^{(1)T}[M]\vec{x}(0) = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.659 \\ 0.183/L \end{Bmatrix}^T \begin{bmatrix} 2m & \frac{mL}{2} \\ \frac{mL}{2} & \frac{mL^2}{3} \end{bmatrix} \begin{Bmatrix} A \\ 0 \end{Bmatrix} = (1.41)A\sqrt{m}$$

$$c_2 = \hat{\vec{X}}^{(2)T}[M]\vec{x}(0) = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.605 \\ -2.18/L \end{Bmatrix}^T \begin{bmatrix} 2m & \frac{mL}{2} \\ \frac{mL}{2} & \frac{mL^2}{3} \end{bmatrix} \begin{Bmatrix} A \\ 0 \end{Bmatrix} = (0.118)A\sqrt{m}$$

$$s_j = \hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0) = 0 \quad ; \quad j = 1,2$$

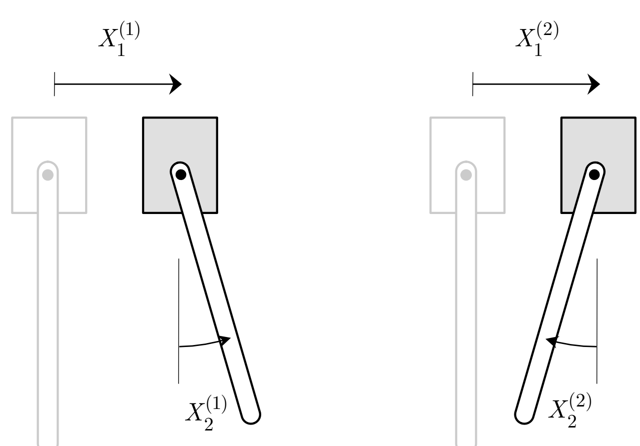
Therefore,

$$\vec{x}(t) = \hat{\vec{X}}^{(1)} c_1 cos\omega_1 t + \hat{\vec{X}}^{(2)} c_2 cos\omega_2 t$$

$$= 1.41A\begin{Bmatrix} 0.659 \\ 0.183/L \end{Bmatrix} cos\left(0.684\sqrt{\frac{k}{m}}t\right) + 0.118A\begin{Bmatrix} 0.605 \\ -2.18/L \end{Bmatrix} cos\left(2.27\sqrt{\frac{k}{m}}t\right)$$

Note that the second mode makes less than 10% of the contribution of the first mode; that is, the first mode dominates the response.

Shown below are the shapes for the two modal vectors. Can envision how this system moves in time?



FIRST mode SECOND mode