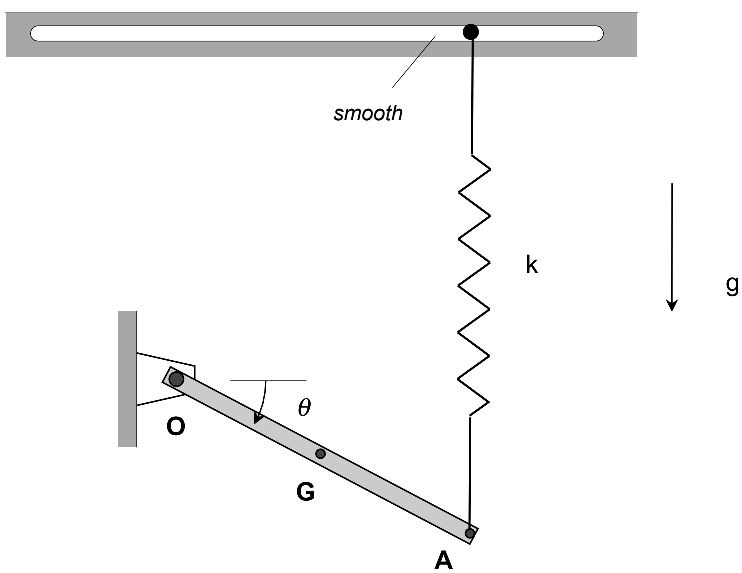
Example A1.11

Given: The thin homogeneous bar shown has a length of L and mass of m. A spring is attached to end A of the bar, with the other end of the spring able to slide in a smooth horizontal slot. Note that having the spring slide in the slot permits the spring to remain vertical for all motion. The spring is unstretched when $\theta = 0$.

Find: For this problem:
a) Determine the equilibrium angle θ_0 for the system. You will find multiple equilibrium angles, including $\pm 90°$ corresponding to the vertical positions of the bar. Consider only the non-vertical equilibrium angle.
b) Determine the linearized EOM of the system for small motion $z = \theta - \theta_0$.
c) What is the natural frequency for small oscillations about the equilibrium state corresponding to $mg/kL = 1.2$?



$$T = \frac{1}{2} I_0 \dot{\theta}^2 \quad ; \; I_0 = I_G + m\left(\frac{L}{2}\right)^2 = \frac{1}{12} mL^2 + \frac{1}{4} mL^2 = \frac{1}{3} mL^2$$

$$U = \frac{1}{2} k (L \sin\theta)^2 - mg \frac{L}{2} \sin\theta$$

(a) <u>For equilibrium</u>:

$$\frac{\partial U}{\partial \theta} = kL^2 \sin\theta \cos\theta - mg \frac{L}{2} \cos\theta = 0$$

$$\hookrightarrow kL^2 \left(\sin\theta - \frac{1}{2} \frac{mg}{kL} \right) \cos\theta = 0$$

- $\cos\theta_0 = 0 \Rightarrow \theta_0 = \pm \frac{\pi}{2}$
- $\sin\theta_0 - \frac{1}{2} \frac{mg}{kL} = 0 \Rightarrow \sin\theta_0 = \frac{1}{2} \frac{mg}{kL} = 0.6$

$$\hookrightarrow \theta_0 = 36.87^\circ$$

(b) Method #1: Apply Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \frac{d}{dt}\left[I_0 \dot\theta\right] = I_0 \ddot\theta$$

$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial U}{\partial \theta} = kL^2\left(\sin\theta - \frac{1}{2}\frac{mg}{kL}\right)\cos\theta$$

$$\therefore \quad I_0 \ddot\theta + \underbrace{kL^2\left(\sin\theta - \frac{1}{2}\frac{mg}{kL}\right)\cos\theta}_{f(\theta)} = 0$$

For static equilibrium:

$$f(\theta_0) = 0 \Rightarrow \left(\sin\theta_0 - \frac{1}{2}\frac{mg}{kL}\right)\cos\theta_0 = 0$$

Either: $\cos\theta_0 = 0 \Rightarrow \theta_0 = \pm\pi/2$

or : $\sin\theta_0 = \frac{1}{2}\frac{mg}{kL} \Rightarrow \sin^2\theta_0 = \frac{1}{4}\left(\frac{mg}{kL}\right)^2$

$$\Rightarrow \cos^2\theta_0 = 1 - \frac{1}{4}\left(\frac{mg}{kL}\right)^2$$

For Taylor series expansion about $\theta = \theta_0$:

$$f(\theta) = \cancelto{0}{f(\theta_0)} + \left.\frac{df}{d\theta}\right|_{\theta_0}(\theta-\theta_0) + \dots$$

$$= kL^2\Big[\underbrace{\cos^2\theta_0 - \sin^2\theta_0}_{1-\frac{1}{2}\left(\frac{mg}{kL}\right)^2} + \underbrace{\frac{1}{2}\frac{mg}{kL}\sin\theta_0}_{\frac{1}{4}\left(\frac{mg}{kL}\right)^2}\Big](\theta-\theta_0)$$

$$= kL^2\left[1 - \frac{1}{4}\left(\frac{mg}{kL}\right)^2\right](\theta-\theta_0) + \dots$$

$\therefore$ Linearized EOM in terms of $z = \theta - \theta_0$ becomes:

$$\frac{mL^2}{3}\ddot z + kL^2\left[1 - \frac{1}{4}\left(\frac{mg}{kL}\right)^2\right] z = 0$$

$$\hookrightarrow \quad \ddot z + \underbrace{\frac{3k}{m}\left[1 - \frac{1}{4}\left(\frac{mg}{kL}\right)^2\right]}_{\omega_n^2 \Rightarrow \omega_n = \sqrt{1-0.6^2}\sqrt{3}\sqrt{\frac{k}{m}} = 0.8\sqrt{2}\sqrt{\frac{k}{m}}} z = 0$$

Method 1a: using $z = \theta - \theta_0 \Rightarrow \theta = z + \theta_0$

$$f(\theta) = kL^2\left[\sin\theta - \frac{1}{2}\frac{mg}{kL}\right]\cos\theta$$

$$= kL^2\left[\sin\theta\cos\theta - \frac{1}{2}\frac{mg}{kL}\cos\theta\right]$$

where:

trig. identities

$$\begin{cases} \sin\theta = \sin(z+\theta_0) \\ \quad = \underbrace{\cos z}_{\approx 1}\sin\theta_0 + \underbrace{\sin z}_{\approx z}\cos\theta_0 = \sin\theta_0 + z\cos\theta_0 + \dots \\ \cos\theta = \cos(z+\theta_0) \\ \quad = \underbrace{\cos z}_{\approx 1}\cos\theta_0 - \underbrace{\sin z}_{z}\sin\theta_0 = \cos\theta_0 - z\sin\theta_0 + \dots \\ \sin\theta\cos\theta = (\sin\theta_0 + z\cos\theta_0 + \dots)(\cos\theta_0 - z\sin\theta_0 + \dots) \\ \quad = \sin\theta_0\cos\theta_0 + (\cos^2\theta_0 - \sin^2\theta_0)z + \dots \end{cases}$$

$$\therefore f(\theta) = kL^2\Big[\sin\theta_0\cos\theta_0 + (\cos^2\theta_0 - \sin^2\theta_0)z - \frac{1}{2}\frac{mg}{kL}(\cos\theta_0 - z\sin\theta_0) + \dots\Big]$$

$$= kL^2\Big[\underbrace{\Big(\sin\theta_0 - \frac{1}{2}\frac{mg}{kL}\Big)\cos\theta_0}_{f(\theta_0)=0} + \Big(\cos^2\theta_0 - \sin^2\theta_0 + \frac{1}{2}\frac{mg}{kL}\sin\theta_0\Big)z + \dots\Big]$$

$$= kL^2\Big[1 - \frac{1}{4}\Big(\frac{mg}{kL}\Big)^2\Big]z + \dots$$

(Same as before)

Method #2

$$K = \frac{\partial^2 U}{\partial\theta^2}\bigg|_{\theta_0} = \frac{d}{d\theta}\Big[kL^2\Big(\sin\theta\cos\theta - \frac{1}{2}\frac{mg}{kL}\cos\theta\Big)\Big]_{\theta_0}$$

$$= kL^2\Big[\cos^2\theta_0 - \sin^2\theta_0 + \frac{1}{2}\frac{mg}{kL}\sin\theta_0\Big]$$

$$= kL^2\Big[1 - \frac{1}{4}\Big(\frac{mg}{kL}\Big)^2\Big]$$

$$\therefore \frac{mL^2}{3}\ddot{z} + kL^2\Big[1 - \frac{1}{4}\Big(\frac{mg}{kL}\Big)^2\Big]z = 0$$

(Same as before)