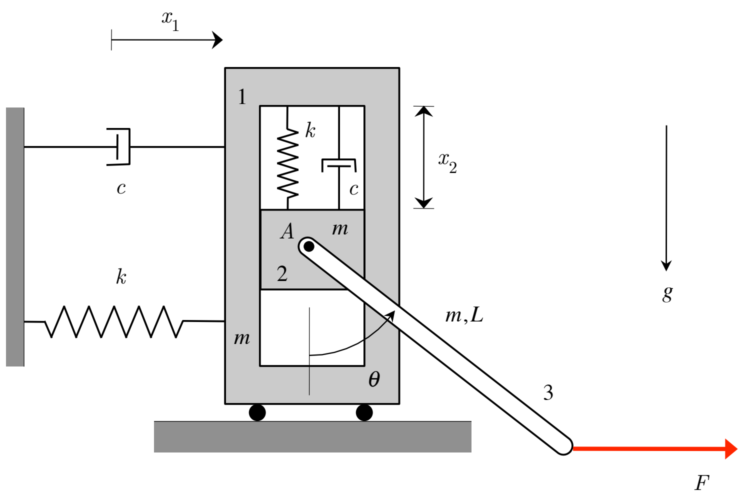
Example A1.10

Given: The system shown is made of bodies 1, 2 and 3, with each body having a mass of m. Body 1 is constrained to move along a smooth horizontal floor. Body 2 is constrained to move within a vertical slot in body 1. Body 3 (a thin, homogeneous bar) is pinned to body 2 at its end A. The coordinates x_1, x_2 and θ are used to describe the position and orientation of the bodies in the system. x_1 is an absolute coordinate, x_2 describes the motion of 2 relative to 1, and θ measures the rotation of body 3 from its downward orientation.

Find: Use Lagrange?s equations to derive the EOMs for this system in terms of the coordinates x_1, x_2 and θ.



SOLUTION

$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 + \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \dot{\theta}^2 \quad \text{where } I_G = \frac{1}{12} m L^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 - mg \frac{L}{2} cos\theta$$

$$R = \frac{1}{2} c \dot{x}_1^2 + \frac{1}{2} c \dot{x}_2^2$$

$$dW = (F\hat{i}) \bullet d\vec{r}_B$$

Kinematics

$$\vec{v}_1 = \dot{x}_1 \hat{i}$$

$$\vec{v}_2 = \vec{v}_1 + \vec{v}_{2/1} = \dot{x}_1 \hat{i} - \dot{x}_2 \hat{j}$$

$$\vec{v}_G = \vec{v}_2 + \vec{v}_{G/A} = \dot{x}_1 \hat{i} - \dot{x}_2 \hat{j} + (\dot{\theta}\hat{k}) \times \left(\frac{L}{2} sin\theta \hat{i} - \frac{L}{2} cos\theta \hat{j} \right)$$

$$= \left(\dot{x}_1 + \frac{L}{2} cos\theta \dot{\theta} \right) \hat{i} + \left(-\dot{x}_2 + \frac{L}{2} sin\theta \dot{\theta} \right) \hat{j}$$

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} = \left(x_1\hat{i} - x_2\hat{j}\right) + \left(Lsin\theta\hat{i} - Lcos\theta\hat{j}\right)$$
$$= \left(x_1 + Lsin\theta\right)\hat{i} + \left(-x_2 - Lcos\theta\right)\hat{j} \quad \Rightarrow$$
$$d\vec{r}_B = \left(dx_1 + Lcos\theta d\theta\right)\hat{i} + \left(-dx_2 + Lsin\theta d\theta\right)\hat{j}$$

Therefore,

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{1}{2}m\left[\left(\dot{x}_1 + \frac{L}{2}cos\theta\dot{\theta}\right)^2 + \left(-\dot{x}_2 + \frac{L}{2}sin\theta\dot{\theta}\right)^2\right] + \frac{1}{2}I_G\dot{\theta}^2$$
$$= \frac{1}{2}(3m)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}m\frac{L^2}{4}(cos^2\theta + sin^2\theta)\dot{\theta}^2 + \frac{1}{2}\left(m\frac{L}{2}cos\theta\right)\dot{x}_1\dot{\theta}$$
$$+ \frac{1}{2}\left(-m\frac{L}{2}sin\theta\right)\dot{x}_2\dot{\theta} + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$
$$= \frac{1}{2}(3m)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}\left(m\frac{L^2}{3}\right)\dot{\theta}^2 + \left(m\frac{L}{4}cos\theta\right)\dot{x}_1\dot{\theta} + \left(-m\frac{L}{4}sin\theta\right)\dot{x}_2\dot{\theta}$$
$$dW = (F\hat{i})\bullet\left[\left(dx_1 + Lcos\theta d\theta\right)\hat{i} + \left(-dx_2 + Lsin\theta d\theta\right)\hat{j}\right]$$
$$= (F)dx_1 + (FLcos\theta)d\theta = Q_{x_1}dx_1 + Q_\theta d\theta$$

Applying Lagrange’s equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left[3m\dot{x}_1 + m\frac{L}{4}cos\theta\dot{\theta}\right] = 3m\ddot{x}_1 + m\frac{L}{4}cos\theta\ddot{\theta} - m\frac{L}{4}sin\theta\dot{\theta}^2$$
$$\frac{\partial T}{\partial x_1} = 0$$
$$\frac{\partial U}{\partial x_1} = kx_1$$
$$\frac{\partial R}{\partial \dot{x}_1} = c\dot{x}_1$$
$$Q_{x_1} = F$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{d}{dt}\left[2m\dot{x}_2 - m\frac{L}{4}sin\theta\dot{\theta}\right] = 2m\ddot{x}_2 - m\frac{L}{4}sin\theta\ddot{\theta} - m\frac{L}{4}cos\theta\dot{\theta}^2$$
$$\frac{\partial T}{\partial x_2} = 0$$
$$\frac{\partial U}{\partial x_2} = kx_2$$
$$\frac{\partial R}{\partial \dot{x}_2} = c\dot{x}_2$$
$$Q_{x_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[\frac{1}{3}mL^2\dot{\theta} + m\frac{L}{4}cos\theta\dot{x}_1 - m\frac{L}{4}sin\theta\dot{x}_2\right]$$

$$= \frac{1}{3}mL^2\ddot{\theta} + m\frac{L}{4}cos\theta\ddot{x}_1 - m\frac{L}{4}sin\theta\ddot{x}_2 - m\frac{L}{4}sin\theta\dot{x}_1\dot{\theta} - m\frac{L}{4}cos\theta\dot{x}_2\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = -m\frac{L}{4}sin\theta\dot{x}_1\dot{\theta} - m\frac{L}{4}cos\theta\dot{x}_2\dot{\theta}$$

$$\frac{\partial U}{\partial \theta} = mg\frac{L}{2}sin\theta$$

$$\frac{\partial R}{\partial \dot{\theta}} = 0$$

$$Q_\theta = FLcos\theta$$