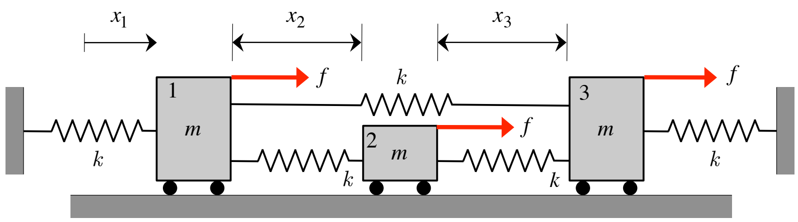
Example A1.9

Given: Three blocks, each of mass m, are able to move along a smooth horizontal surface. The blocks are interconnected by springs, as shown in the figure. Identical forces f act to the right on each of the blocks. Let $x_1$represent the absolute motion of block 1, x_2 represent the motion of 2 relative to 1 and x_3 represent the motion of 3 relative to 2.

Find: Use Lagrange?s equations to derive the EOMs for this three-DOF system in terms of the generalized coordinates x_1, x_2 and x_3.



SOLUTION

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1+\dot{x}_2\right)^2 + \frac{1}{2}m\left(\dot{x}_1+\dot{x}_2+\dot{x}_3\right)^2 \\
&= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2\right) \\
&\qquad + \frac{1}{2}m\left[\dot{x}_1^2 + 2\dot{x}_1\left(\dot{x}_2+\dot{x}_3\right) + \dot{x}_2^2 + 2\dot{x}_2\dot{x}_3 + \dot{x}_3^2\right] \\
&= \frac{1}{2}(3m)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}(m)\dot{x}_3^2 + \frac{1}{2}(4m)\dot{x}_1\dot{x}_2 + \frac{1}{2}(2m)\dot{x}_1\dot{x}_3 + \frac{1}{2}(2m)\dot{x}_2\dot{x}_3 \\
&= \frac{1}{2}(m_{11})\dot{x}_1^2 + \frac{1}{2}(m_{22})\dot{x}_2^2 + \frac{1}{2}(m_{33})\dot{x}_3^2 + \frac{1}{2}(m_{12}+m_{21})\dot{x}_1\dot{x}_2 \\
&\qquad + \frac{1}{2}(m_{13}+m_{31})\dot{x}_1\dot{x}_3 + \frac{1}{2}(m_{23}+m_{32})\dot{x}_2\dot{x}_3
\end{aligned}
$$

$$
U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}kx_3^2 + \frac{1}{2}k(x_2+x_3)^2 + \frac{1}{2}k(x_1+x_2+x_3)^2
$$

$$
\begin{aligned}
dW &= (f\hat{i})\bullet d\vec{r}_1 + (f\hat{i})\bullet d\vec{r}_2 + (f\hat{i})\bullet d\vec{r}_3 \\
&= (f\hat{i})\bullet d(x_1\hat{i}) + (f\hat{i})\bullet d(x_1+x_2)\hat{i} + (f\hat{i})\bullet d(x_1+x_2+x_3)\hat{i} \\
&= (3f)dx_1 + (2f)dx_2 + (f)dx_3 = Q_1dx_1 + Q_2dx_2 + Q_3dx_3
\end{aligned}
$$

Applying Lagrange's equations gives:

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0 \quad \Rightarrow \quad 3m\ddot{x}_1 + 2m\ddot{x}_2 + m\ddot{x}_3 + 2kx_1 + kx_2 + kx_3 = 3f
$$

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0 \quad \Rightarrow \quad 2m\ddot{x}_1 + 2m\ddot{x}_2 + m\ddot{x}_3 + kx_1 + 3kx_2 + 2kx_3 = 2f
$$

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) - \frac{\partial T}{\partial x_3} + \frac{\partial U}{\partial x_3} = 0 \quad \Rightarrow \quad m\ddot{x}_1 + m\ddot{x}_2 + m\ddot{x}_3 + kx_1 + 2kx_2 + 3kx_3 = f
$$

In matrix form, these EOMs become:

$$
\begin{bmatrix} 3m & 2m & m \\ 2m & 2m & m \\ m & m & m \end{bmatrix}
\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}
+
\begin{bmatrix} 2k & k & k \\ k & 3k & 2k \\ k & 2k & 3k \end{bmatrix}
\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}
=
\begin{Bmatrix} 3f \\ 2f \\ f \end{Bmatrix}
$$

Alternately, we can use the explicit description of our mass, damping and stiffness matrices (from the section of linearization of EOMs):

$$
[M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}
\qquad
[K] = \left[\frac{\partial^2 U}{\partial \dot{q}_i \partial \dot{q}_j}\right]_{\vec{0}}
$$

to arrive at the same results for the mass and stiffness matrices.