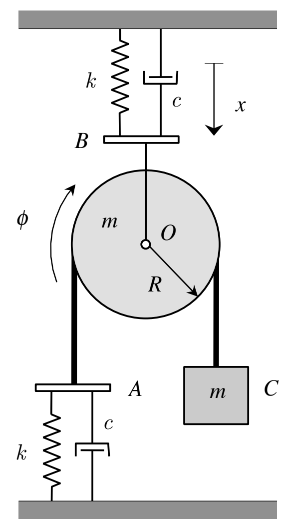
Example A1.8

Given: A homogeneous disk of mass m and outer radius R is supported by an in-parallel connection of a spring (of stiffness k) and of a dashpot (of damping coefficient c). An inextensible cable is wrapped around the outer perimeter of the disk. One end of the cable is attached to a second, in-parallel spring/dashpot connection, with the other end attached to block C (of mass m). Let x represent the motion of the massless connector B, and ϕ the rotation of the disk. Let $\phi = 0$ when the springs are unstretched. Assume that the cable does not slip on the disk. All motion of the system occurs in a horizontal plane.

Find: Use Lagrange's equations to derive the EOM for this single-DOF system in terms of the generalized coordinate ϕ.



SOLUTION

$$T = T_{disk} + T_{block} = \frac{1}{2}mv_O^2 + \frac{1}{2}I_O\dot{\phi}^2 + \frac{1}{2}mv_C^2$$

$$U = \frac{1}{2}k\Delta_A^2 + \frac{1}{2}k\Delta_B^2$$

$$R = \frac{1}{2}c\dot{\Delta}_A^2 + \frac{1}{2}c\dot{\Delta}_B^2$$

Kinematics

Let C' be the point on the right side of the disk from where the cable comes away:

$$\begin{aligned}\vec{v}_{C'} &= \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O} \\ &= -\dot{x}\hat{j} + \left(-\dot{\phi}\hat{k}\right) \times (R\hat{i}) \\ &= -\left(\dot{x} + R\dot{\phi}\right)\hat{j} = \vec{v}_C\end{aligned}$$

$$\Delta_A = x - R\phi \quad \Rightarrow \quad \dot{\Delta}_A = \dot{x} - R\dot{\phi}$$

$$\Delta_B = x \quad \Rightarrow \quad \dot{\Delta}_B = \dot{x}$$

Therefore, we have:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\phi}^2 + \frac{1}{2}m\left(\dot{x}+R\dot{\phi}\right)^2$$

$$= \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{3}{2}mR^2\right)\dot{\phi}^2 + \frac{1}{2}(2mR)\dot{x}\dot{\phi}$$

$$= \frac{1}{2}m_{11}\dot{x}^2 + \frac{1}{2}m_{22}\dot{\phi}^2 + \frac{1}{2}(m_{12}+m_{21})\dot{x}\dot{\phi}$$

$$U = \frac{1}{2}k\left(x-R\phi\right)^2 + \frac{1}{2}kx^2 = \frac{1}{2}(2k)x^2 + \frac{1}{2}(kR^2)\phi^2 - (kR)x\phi$$

$$R = \frac{1}{2}c\left(\dot{x}-R\dot{\phi}\right)^2 + \frac{1}{2}c\dot{x}^2 = \frac{1}{2}(2c)\dot{x}^2 + \frac{1}{2}(cR^2)\dot{\phi}^2 - (cR)\dot{x}\dot{\phi}$$

Applying Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}\left[2m\dot{x} + mR\dot{\phi}\right] = 2m\ddot{x} + mR\ddot{\phi}$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial U}{\partial x} = 2kx - kR\phi$$

$$\frac{\partial R}{\partial \dot{x}} = 2c\dot{x} - cR\dot{\phi}$$

and:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left[mR\dot{x} + \frac{3}{2}mR^2\dot{\phi}\right] = mR\ddot{x} + \frac{3}{2}mR^2\ddot{\phi}$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial U}{\partial \phi} = -kRx + kR^2\phi$$

$$\frac{\partial R}{\partial \dot{\phi}} = -kR\dot{x} + kR^2\dot{\phi}$$

Together, these give the following two EOMs:

$$2m\ddot{x} + mR\ddot{\phi} + 2c\dot{x} - cR\dot{\phi} + 2kx - kR\phi = 0$$

$$mR\ddot{x} + \frac{3}{2}mR^2\ddot{\phi} - kR\dot{x} + kR^2\dot{\phi} - kRx + kR^2\phi = 0$$

or, in matrix form:

$$\begin{bmatrix} 2m & mR \\ mR & 3mR^2/2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} 2c & -cR \\ -cR & cR^2 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} 2k & -kR \\ -kR & kR^2 \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Alternately, we can use the explicit description of our mass, damping and stiffness matrices (from the section of linearization of EOMs):

$$[M]=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}=\begin{bmatrix} 2m & mR \\ mR & 3mR^2/2 \end{bmatrix} \text{ (see expression for KE above)}$$

$$[C]=\left[\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j}\right]_{\vec{0}}=\begin{bmatrix} 2c & -cR \\ -cR & cR^2 \end{bmatrix}$$

$$[K]=\left[\frac{\partial^2 U}{\partial \dot{q}_i \partial \dot{q}_j}\right]_{\vec{0}}=\begin{bmatrix} 2k & -kR \\ -kR & kR^2 \end{bmatrix}$$

These agree with the above derivation that directly uses Lagrange's equations, as one would expect.