An object-oriented implementation of the SPIKE preconditioner in MEMOSA-FVM
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Overview
The SPIKE algorithm was originally devised as a parallel solver for banded linear systems. The method has been extended to work as a preconditioner for general sparse systems using the following approach.

(i) The Fiedler vector for the given matrix is computed and sorted to give the Fiedler permutation. This permutation is applied to the rows and columns of the matrix.

(ii) A banded matrix with a user-specified bandwidth is extracted from the re-ordered matrix. This is used as a preconditioner within a Krylov subspace method such as BICGSTab.

(iii) The parallel Spike algorithm is used to apply the preconditioner within BICGSTab.

We have implemented a Spike solver class with the MEMOSA-FVM framework.

The Fiedler permutation
Let A be a symmetric matrix of order n. The weighted Laplacian matrix L for A is given by:

\[
\begin{align*}
L(i,i) &= \sum_k |A(i,k)|, \quad \text{for } k = 1, 2, \ldots, n; \ k \neq i \\
L(i,j) &= -|A(i,j)|, \quad \text{for } i \neq j
\end{align*}
\]

Obtain the eigenvector corresponding to the second smallest eigenvalue of L:

\[
L x_2 = \lambda_2 x_2
\]

Sort the Fiedler vector \(x_2\) based on the values of its entries, to obtain the Fiedler permutation.

We compute the Fiedler vector using the Trace Minimization parallel eigenvalue solver developed at Purdue.

Metis and Fiedler Orderings in 2D
Metis partitioning shown for a mesh and the corresponding matrix. The matrix is shown in the original ordering, with the Metis partitions shown in color code. Fiedler reordering shown for the same mesh and the corresponding matrix. The matrix is shown after the Fiedler reordering.

Metis and Fiedler Orderings in 3D
The top two figures are two views of the Metis partitioning for a 3D mesh. The bottom two figures are two views of the Fiedler partitioning for the same mesh.

Summary
We have integrated the SPIKE preconditioner into the object-oriented framework of MEMOSA-FVM. We have demonstrated excellent scalability of the SPIKE preconditioner in conjunction with BICGSTab for the momentum and continuity equations for flow problems. The SPIKE preconditioner has the same templating capability as FVM.

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The Spike Algorithm
\[
A = D \cdot S \\
D = \text{diag}(A_1, A_2, \ldots, A_p)
\]

The Spike Matrix “S”

Reduced System

Future Work
The performance of the SPIKE preconditioner will be further enhanced by incorporating additional solvers for the diagonal blocks, and by optimizing the computation of the tips of the spikes that are part of the reduced system in the SPIKE matrix.

Scalability
SPIKE preconditioner with semi-bandwidth=10, applied with the BCGSTAB solver to two hexahedral meshes.

Convergence to run 100 iterations with the linear solver tolerance = 1e-1 for the continuity and 1e-3 for the momentum equation.

4 cores per node on the Hera cluster at LLNL, using MVAPICH2.1.4.1.