Quality Factor for Cantilever Beams

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Assumptions:
- Gas Damping at Low Pressure
- Edges and tip of beam neglected
- Gas velocity always normal to surface
- Gas and beam temperatures are the same
- Gas damping is proportional to the beam velocity:
  \[ Dv_{\text{normal}} = \frac{P_{\text{ambient}}}{(4 + \pi) \frac{1}{2} \pi \rho \omega^2} \]  
  (Valid since beam velocity is small compared to thermal velocity of gas.)

Beam theory for vertical displacement \( u(x,t) \):

\[
\rho bh u_{tt} + E I u_{xxxx} + D bu_t = 0
\]
\[
u_{tt} + c^2 u_{xxxx} + 2\sigma u_t = 0
\]

Solve by separation of variables \( u(x,t) = X(x)T(t) \) with non-negative separation constant \( \lambda \)

\[ \frac{T_{tt} + 2\sigma T_t}{T} = -c^2 \frac{X_{xxxx}}{X} = -\lambda \]

Spatial solution gives modes of vibration.
Temporal term is identical to single DOF damped spring-mass system:

\[ T_{tt} + 2\sigma T_t + \lambda T = 0 \]

Definition of Quality factor for single DOF system:

\[ Q = \frac{\omega_0}{2\sigma} \]

Definition of Quality factor for each mode of a cantilever beam:

\[ Q = \frac{\omega_{0,n}}{2\sigma} \frac{\rho h}{D} \]

Remove zero-pressure decay to correct Quality factor:

\[ Q_{\text{corr}} = \frac{\omega_{0,n}}{2(\sigma_{\text{raw}} - \sigma_{\text{pres}=0})} \]

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- Gas Damping at Low Pressure
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\[ P_{\text{normal}} = -Dv \]

\[ D = P_{\text{ambient}}(4 + \pi) \sqrt{\frac{1}{2\pi RT}} \]

Method:
- Solve MPM momentum equation
- Record beam-tip vertical velocity each time step.
- Fit tip velocity \( u(t) \) to a decaying sinusoid using a nonlinear Gauss-Newton algorithm to find the parameters \( A, s, w, f \).
- Calculate \( Q = \omega/2\sigma \) and \( Q_{\text{corr}} \).

Discrete MPM Momentum Equation:

\[ m_i v_{i+1} = m_i v_i + [f_i^{\text{int}} - m_i v_i D/(\rho h)] \Delta t \]

at each node \( i \)

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Material-Point Method (MPM) Simulations

Probe Chip: Beam A

http://spmtips.com/csc/c12/tipless

Top view of cantilever beam undergoing mode 1 vibration. Material points colored by velocity

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E = Young’s Modulus

\( \rho \) = density

b = beam width

L = beam length

I = bh/12

\( D \) = constant damping factor

\( 2\sigma = D/(\rho h) \)