Distributional Sensitivity Analysis
Akil Narayan, Dongbin Xiu
Purdue University

UQ for a stochastic system

\[ \mathbb{L} \text{ is a model of a physical system: } \mathbb{L}(u, x, z) = 0 \]

- \( x \) are spatial/temporal variables
- \( u \) is a modeled quantity
- \( z \) are random parameters with density \( \rho \).

Often we are interested not in \( u \), but in some QoI \( \xi \):

\[ \xi = \int_\Omega f(u) \rho(z) dz \]

Examples:

\[ f(u) = u \Rightarrow \xi \text{ is the mean of } u \]
\[ f(u) = \|u - \int_\Omega u \rho(z) dz\|^2 \Rightarrow \xi \text{ is the variance of } u \]

Concern: how does the distribution of \( z \) affect \( u \) or \( \xi \)?

SA: how do the \textit{values} of the random parameters \( z \) affect the solution and QoI?

The model approximation \( u_{\rho} \) from (1) can be used to compute e.g. sensitivity derivatives \( \frac{\partial \xi}{\partial \rho} \).

Known result: \textit{Convergence of the approximated sensitivity to the true sensitivity is as fast as convergence of the model solution }\( u_{\rho} \) \textit{to the true solution.}

This is an \textit{aleatory} result: we assume all information about how \( z \) is distributed is known; in practice such information is rarely present.

Concern: how sensitive is \( \xi \) to the density \( \rho \)?

DSA deals with the \textit{epistemic uncertainty} of an unknown probability distribution for \( z \). For two densities \( \rho, \sigma \), the definition of Distributional Sensitivity for the QoI \( \xi(u) \):

\[ DS[\rho, \sigma](u) = \frac{\|\xi_{\rho}(u_{\rho}) - \xi_{\sigma}(u_{\sigma})\|}{\text{dist}(\rho, \sigma)} \]

Challenge in computing the DS: we require \( \xi_{\rho}(u_{\rho}) \) and therefore the model solution \( u_{\rho} \Rightarrow \) more simulations/experiments of (1) must be performed. To avoid this, we can instead use \( u_{\rho} \) as a surrogate.

Theorem (Narayan, Xiu): When \( \xi \) is the mean or variance, then convergence of the surrogate DS approximation to the true DS is as fast as convergence of the surrogate \( u_{\rho} \) to \( u \).

Result: DSA is a post-processing step (therefore nonintrusive) and \textit{requires no additional model simulations/experiments}.

Right: \( \xi \) is the location of the transition layer in a Burgers' equation model, a convection-diffusion system with random boundary conditions. Level sets of the DS are shown. Assuming a uniform density in the model simulation, the DS is computed against boundary conditions that are distributed as random variables \( \sim \text{Beta}(\alpha, \beta) \).

Left: \( \xi \) is the mean of a one-dimensional diffusion process with random diffusion, the density of which is characterized by \( \alpha \). The DS magnitude is shown as a function of locations \( z \).